Virtual contraction and passivity based control of nonlinear mechanical systems

Reyes Báez, Rodolfo

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Virtual Contraction and Passivity based Control of Nonlinear Mechanical Systems

Trajectory Tracking and Group Coordination

Rodolfo Reyes-Báez

The research described in this dissertation has been carried out at the Bernoulli Institute of Mathematics, Computer Science and Artificial Intelligence, Faculty of Science and Engineering of the University of Groningen, in Groningen The Netherlands.

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Cover[1]: *"Quetzalcoatl and Kukulkan are mutually virtual systems"*
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[1]The Aztec God Quetzalcoatl and the Mayan God Kukulkan represent the Feathered Serpent deity of many Mesoamerican religions in the Nahuatl and Mayan languages, respectively (Miller and Taube 1997).

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Prof.dr.ir. B. Jayawardhana

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Prof.dr.ir. J.M.A. Scherpen
Prof.dr. R. Wisniewski
Prof.dr. I.R. Manchester

To my beloved parents
Eloísa Báez and Demetrio Reyes,
and my brothers
Marcos Emilio and Juan José;

and

... to the memory of my uncles
Francisco Medel and Ernesto Báez,
who always challenged me to go further.

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[2]Er gaat niets boven Groningen!

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Last paragraph in spanish:

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Rodolfo Reyes-Báez
Groningen, The Netherlands
August 19, 2019

Contents

List of symbols and acronyms

$F_{N_v}(q,\dot{q},\dot{q}_v)$	Virtual force map $T\mathcal{X}\times T\mathcal{X}\to T^*\mathcal{X}$ locally induced by $N(q,\dot{q})$	28
pH	acronym of port-Hamiltonian	33
H	Hamiltonian function $H\in C^\infty(\mathcal{X})$, or total energy	33
P	Potential energy function $H\in C^\infty(\mathcal{X})$	33
$E(q,p)$	Hamiltonian counterpart of the Lagrangian Coriolis matrix $C(q,\dot{q})$	34
$\mathbb{F}(q,p)$	Hamiltonian counterpart of the force map $F_N(q,\dot{q})$	35
$\overline{\mathbb{F}}(q,p)$	Hamiltonian counterpart of the force map $F_{N_v}(q,\dot{q},\dot{q}_v)$	36
$\mathcal{J}_v(q,p)$	Structure matrix of an almost-Poisson bracket	40
u^{ff}	Feedforward control action	45
u^{fb}	Feedback control action	45
$\Omega(t)$	Sliding manifold with sliding variable $\sigma(x,t)$	52
$FJRs$	acronym of Flexible Joints Robots	63
$SNAME$	acronym of Society of Naval Architects and Marine Engineers	88
p^b	Quasi-momentum in the body-fixed frame $\{b\}$	94
PBC	acronym of Passivity Based Control	109

Chapter 1

Introduction

> ”A good paper should contain at least one serious error, in order to add some magic to it”
>
> -Jan C. Willems (Paraphrased by A. J. van der Schaft)

In this chapter a general overview of the control problems that are worked in this dissertation are presented. The main results are summarized in the list of publications.

1.1 Literature review

1.1.1 Tracking control of mechanical port-Hamiltonian systems

The control of electro-mechanical (EM) systems is a well-studied problem in systems and control literature. Many control design tools have been proposed and studied to solve the stabilization problem using the Euler-Lagrange (EL) formalism for describing the dynamics of EM systems. The physical structure of the EL system is exploited through passivity-based control (PBC) methods which are expounded in (Ortega et al. 2013) and references therein. These techniques were extended to solve the problem of motion control of EM (which includes trajectory tracking and path-following) using the EL formalism since the resulting control schemes have a clear physical interpretation in terms of *co-energy* variables. The interested reader on the early work of tracking control for EL systems is referred to (Slotine and Li 1987) and (Kelly et al. 2006, Jayawardhana and Weiss 2008).

As an alternative to the EL formalism, the port-Hamiltonian (pH) framework has been proposed (see the pioneering work of (van der Schaft and Maschke 1995a)), which is a rather elegant and practical approach for analysis and design of (nonlinear) control systems. Among the main characteristics of the pH framework we have the following: *i*) the existence of a Dirac structure, which connects the underlying state space geometry with the system's analysis tools, this by taking the Hamiltonian function as a Lyapunov function; *ii*) it provides a port-based network modeling that enables open systems modeling through dissipativity theory. These two characteristics let the pH framework to have a clear physical interpretation in terms of *energy* variables, since the energy function can directly be used to show the dissipativity and stability properties of the systems. The port-based modeling of pH systems is modular in the sense that if two pH systems are interconnected through their external ports with a *power-preserving* interconnection, then the resulting interconnected system is also a pH system.

A number of set-point control design methods for mechanical pH systems have been proposed during the past two decades. For instance, the standard proportional-integral (PI) control (Jayawardhana et al. 2007), Interconnection and Damping Assignment Passivity based Control (IDA-PBC) (Ortega et al. 2002), Control by Interconnection (CbI) method (Ortega et al. 2008, Ortega and Borja 2014a), PID passivity-based control (Borja-Rosales 2017, Zhang et al. 2018, Romero et al. 2018); among many others. Moreover, several successful industrial implementations of passivity-based controllers in the pH framework have been reported. See for instance (Sepulchre et al. 2013).

Nevertheless, for trajectory tracking control problems it is not straightforward to design controllers for such pH systems with an insightful energy interpretation of the closed-loop system. For example, it is not trivial to obtain an incremental passive system (Jayawardhana 2006) via a controller interconnected with the pH system. A major difficulty is that the external reference signals can induce both the closed-loop system and total energy function to be time-varying. In this case, the usual LaSalle invariance principle can not be invoked for the convergence analysis. In order to overcome this, a structure preserving error system is introduced in (Fujimoto et al. 2003) which is based on generalized canonical transformations (GCTs). Necessary and sufficient conditions for passivity preserving state transformations are given; correspondingly once in the new canonical coordinates, the pH error system can be stabilized with standard passivity-based control.

For mechanical pH systems, in the works of (Dirksz and Scherpen 2010) and (Romero, Donaire, Navarro-Alarcon and Ramirez 2015), a GCT is used to obtain a pH system which is *linear* in the momentum with constant inertia matrix; resulting in a *quasi-linear* system. The tracking control scheme is then proposed to preserve the quasi-linear a pH structure for the closed-loop error system. Although solving partial differential equations that correspond to the existence of such GCT is *not trivial*, some characterizations are presented in (Venkatraman et al. 2010) for specific classes of mechanical systems. Some further extensions of these methods to other classes of systems include the works of (Donaire et al. 2017) for pH systems on moving frames with application to marine craft control, and the work of (Jardón-Kojakhmetov et al. 2016) where a class of *underactuated* mechanical pH systems is considered to solve the tracking problem for flexible joints robots using the singular perturbations approach. Similarly, a tracking controller for pH systems via contraction analysis is presented in (Yaghmaei and Yazdanpanah 2017), where a class of *contractive pH systems* is nicely characterized. These systems are then used in the IDA-PBC method as target dynamics.

It should be noticed that in all the control schemes mentioned above, there may exists non-tractability problems since a set of complex PDEs needs to be solved.

1.1.2 Group coordination of mechanical systems

The use of collaborative robots (which include mobile robots, marine systems and UAVs) and of networked electro-mechanical systems are pervasive in various application domains, such as, smart factories, smart logistic systems, intelligent buildings and smart grids. For instance, the collaborative robots can be deployed to solve a variety of different tasks by autonomously coordinating their movements and actions among themselves. As another example, a network of machines in the shop floor of a smart factory can reconfigure themselves cooperatively and autonomously to produce a variety of different products. Against the backdrop, the distributed control methods thereof have been an active area of research for the past decade, providing control algorithms that can guarantee the completion of every given task by the group of robots or by the networked machines. These physical systems typically belong to classes of systems in the energy-based frameworks; as the ones described in the previous section.

The second part of this work is focused on the distributed (tracking) control of networked mechanical systems in the EL framework, which are a particular class of the so-called *multi-agent systems*. The generalization of the PBC methods described in the previous section (for a single mechanical EL system) to the multi-agent setting has been well-studied in recent decade. The book of (Bai et al. 2011), (van der Schaft 2017) and the articles by (Chopra and Spong 2006) and (Arcak 2007) provide a thorough exposition to the design of passivity-based distributed control where a number of coordination control problems can be solved through PBC approach, including, synchronization and formation control. For networked EL systems, some relevant works are the articles by (Garcia de Marina Peinado et al. 2018), (Nuño, Ortega, Jayawardhana and Basanez 2013), (Nuño, Ortega, Jayawardhana and Basañez 2013) and (Chung and Slotine 2009a). The proposed approach here is also close related to the port-Hamiltonian counterpart presented in (Vos 2015), where motivated by (Arcak 2007), the agents are assumed to be point masses; this is not the case in the present work. Moreover, this approach solves not only the velocity coordination problem of mechanical systems, but also the coordinated position control.

1.1.3 Contraction analsyis and virtual systems

Contraction analysis was introduced to the systems and control community in the work of (Lohmiller and Slotine 1998) as a *differential approach* to incremental stability. Conceptually, a system is called contracting if any pair of neighboring trajectories converge to each other, see (Jouffroy and Fossen 2010) and references therein. Contraction analysis has been studied with different approaches, such as the one in (di Bernardo et al. 2009) using the *matrix or logarithmic measures*, and in (Pavlov and van de Wouw 2017) using the *convergent dynamics* where constant Riemannian metrics are used. A unifying framework is presented in (Forni and Sepulchre 2014) where *Finsler geometry* is employed to develop *Lyapunov-like* conditions in order to analyze contractive behavior.

Contraction analysis is extended to *systems with inputs* in (Sontag 2010) in terms of matrix measures, in (Manchester and Slotine 2014a) for *contraction based control* (CBC) design via contraction metrics, and from a *differential dissipativity* approach in (Manchester and Slotine 2014b, Forni and Sepulchre 2013). This is further explored in (van der Schaft 2015) from a geometric point of view. In particular, the differential Lyapunov is extended to *differential passivity* in (Forni et al. 2013, van der Schaft 2013).

The above concepts of contractivity are further generalized by exploiting the notion of virtual systems in order to infer the convergence behavior of a given original system (Wang and Slotine 2005, Jouffroy and Fossen 2010, Sontag 2010, Forni and Sepulchre 2014). Roughly speaking, for a given plant, a virtual system can be understood as a system that can produce all plan's trajectories, i.e., the plant's *behavior* is *embedded* in the virtual one. Virtual systems are commonly found in state estimation and tracking problems. For instance, in state estimation, the original system is the reference system and the virtual system is the observer itself. If the virtual system is contracting then all of its solutions will converge to any plant's trajectory. This concept is referred[1] to as *virtual contraction*. Analogously, the same idea has been briefly extended to virtual systems with inputs in (Jouffroy and Fossen 2010, Manchester et al. 2018), for what is called *virtual contraction based control* (v-CBC) in this work.

1.2 Contribution of the thesis

The main differences contributions in this dissertation are summarized bellow:

- The definition of virtual control systems is generalized with respect to the one presented in (Wang and Slotine 2005, Jouffroy and Fossen 2010, Sontag 2010, Forni and Sepulchre 2014, van der Schaft 2017). Here, it is allowed that the virtual

[1]This is also referred as *partial contraction* in (Wang and Slotine 2005, Jouffroy and Fossen 2010).

control system input to be different from the original plant's one. Furthermore, the steady-state solution is characterized using convergent dynamics arguments.

- A class of virtual control systems associated to mechanical systems are introduced in both the EL and pH energy-based frameworks. These virtual mechanical systems are constructed in terms of a mathematical object called *virtual force* which under the right assumptions behaves as a *true force*. The structure of the virtual forces is characterized by the underlying geometry of the state space; Riemannian manifolds for EL systems and almost-Poisson manifolds for pH systems. This in turn implies that the virtual systems have a clear physical interpretation in terms of passivity (lossless or energy conservation).

- A *family* of virtual contraction based (tracking) controllers for *fully-actuated* mechanical pH systems is proposed. Sufficient conditions under which the closed-loop system *preserves* the virtual system's structure are given. Existence of an invariant and attractive *sliding manifold* of the closed-loop system is shown. Three different controllers within this approach are constructed for a rigid robot manipulator of two degrees of freedom and experimentally evaluated. It is also shown that each of these controllers exhibit different structural and convergent properties. Among the differences with the related works (Dirksz and Scherpen 2010, Romero, Ortega and Sarras 2015, Romero, Donaire, Navarro-Alarcon and Ramirez 2015) and references therein, it is pointed out that in this work it is not necessary to perform a preliminary change of coordinates in de control design process.

- The tracking control problem of flexible-joints robots (FJRs) modeled as a class of *underactuated* port-Hamiltonian systems is solved using the proposed v-CBC methodology, which is a different approach to the only work on this topic in (Jardón-Kojakhmetov et al. 2016), where the singular perturbation approach is applied. In this work it is not assumed any time-scale separation in the synthesis. Two novel virtual contraction based tracking controllers for FJRs are designed using the Riemannian metric and matrix measure contraction approaches. Similar to the rigid robot case, these two controllers exhibit different *structural properties* such as passivity and differential passivity. The performance is *experimentally validated* on a planar robot with two flexible-joints.

- The proposed v-CBC is also applied to solve the tracking problem of marine craft which are modeled as mechanical systems on moving frames. The introduced concept of virtual forces is then used to propose virtual systems for the existing pH models for marine craft in (Donaire et al. 2017). Two v-CBC schemes for marine craft are designed, one on the body-frame and the other on the inertial one.

The scheme in the inertial frame solves the open problem in (Donaire et al. 2017) without the intermediate change of coordinates.

- The PBC design methodology in (Arcak 2007), where roughly speaking the coordination control problem is solved by interconnecting (strictly passive) systems attached to the nodes of a graph via diffusive coupling that preserves the passivity of the network dynamics, is reformulated. As an alternative, strictly passive *artificial spring systems* are attached to each node and they are feedback interconnected to the nodes dynamics. This results in dynamics protocols where the spring dynamics can be interpreted as a (nonlinear) integral action. Due to the strict passivity of the interconnected system, the asymptotic stability result can be established by using the total storage function as a *strict* Lyapunov function. This then is applied to solve the coordination control problem in EL systems. Two other distributed control methods which use networked strictly passive virtual systems and can be seen as a particular case of our proposed method.

1.3 List of publications

All the publications resulting of this thesis are enlisted below. These are divided in journal papers, conference papers and conference abstracts.

Journal papers

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu Jayawardhana, ”Virtual differential passivity based control for a class of mechanical systems in the port-Hamiltonian framework ” under review, 2019.

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu Jayawardhana, Le Pan, ”A family of virtual contraction based controllers for tracking of flexible joints port-Hamiltonian robots: theory and experiments ” under review, 2019.

- Rodolfo Reyes-Báez, Pablo Borja-Rosales, Arjan van der Schaft, Bayu Jayawardhana, Le Pan, ”On energy-based virtual mechanical systems in the Euler-Lagrange and port-Hamiltonian frameworks”, in preparation.

Conference papers

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu jayawardhana, ”Tracking Control of Fully-actuated port-Hamiltonian Mechanical Systems via Contraction Analysis”, In Proceedings of 20th IFAC World Congress, Toulouse France, 2017.

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu Jayawardhana, ”Virtual Differential Passivity based Control for Tracking of Flexible-joints Robots”, October 2017, 6th IFAC Workshop on Lagrangian and Hamiltonian Methods for Nonlinear Control - LHMNC, Valparaíso Chile, 2018.

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu Jayawardhana, ”Passivity based distributed tracking control of networked Euler-Lagrange systems”, 7th IFAC Workshop on Distributed Estimation and Control in Networked Systems - NECSYS, Groningen The Netherlands, 2018.

- Rodolfo Reyes-Báez, Alejandro Donaire, Arjan van der Schaft, Bayu Jayawardhana, Tristan Pérez, ”Tracking Control of Marine Craft in the port-Hamiltonian Framework: A Virtual Differential Passivity Approach”, European Control Conference, Naples Italy, 2019.

- Rodolfo Reyes-Báez, Pablo Borja, Arjan van der Schaft, Bayu Jayawardhana, ”Virtual mechanical systems: an energy-based approach”, Submitted AMCA National Congress of Automatic Control, Puebla Mexico, 2019.

Conference abstracts

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu Jayawardhana, ”Contraction-based Control Design for Physical Systems”, In Proceedings of 35th Benelux Meeting on Systems and Control, Soesterberg The Netherlands, 2016.

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu Jayawardhana, ”The partial contraction approach for convergence analysis in the tracking control of mechanical port-Hamiltonian systems”, In Proceedings of 36th Benelux Meeting on Systems and Control, Spa Belgium, 2017.

- Rodolfo Reyes-Báez, Arjan van der Schaft, Bayu Jayawardhana, ”Virtual differential passivity based control of mechanical systems in the port-Hamiltonian framework ”, In Proceedings of 37th Benelux Meeting on Systems and Control, Soesterberg The Netherlands, 2018.

Graduations projects

- Le Pan, ”Differential passivity based control of a robotic manipulator”, Master’s thesis, University of Groningen, 2018.

- José Domingo Pájaro-Adrián, ”Experimental evaluation of tracking controllers for robot manipulators: a passive virtual mechanical systems approach (in Spanish)”, Master’s thesis, Autonomous University of Puebla (BUAP), Mexico, in process.
- Lorenzo Lázaro González-Romeo, ”Contraction-based variable gain control with applications in servo-systems (in Spanish)”, Master’s thesis, Autonomous University of Puebla (BUAP), Mexico, in process.

1.4 Outline of the thesis

The outline of the reminder of the thesis is as follows:

In Chapter 2 a self-contained survey of the theoretical preliminaries on contraction analysis, differential passivity and virtual systems used in the thesis are presented. Moreover, in this chapter the *virtual contraction based control* (v-CBC) method is proposed.

Chapter 3 presents the detailed construction of a class of energy-based virtual control systems associated to mechanical systems in the Euler- Lagrange (EL) and port-Hamiltonian (pH) frameworks using the notion of *virtual forces*.

In Chapter 4 the control problem of fully-actuated nonlinear mechanical systems in the port-Hamiltonian framework is solved via the virtual contraction-based control (v-CBC). Closed-loop system properties and experimental evaluation are also presented.

A natural extension of this result is developed in Chapter 5 for flexible-joints robots (FJRs) which are modeled as class of underactuated mechanical pH systems. It is shown that under potential energy matching conditions, the corresponding closed-loop virtual system is contractive.

The results of Chapter 4 are further extended in Chapter 6 to the case of marine craft which are modeled as rigid bodies on moving frames. Due to the controller construction is performed in two scenarios, in a body-fixed and inertial frame.

The problem of coordination control is solved by means of the passivity properties of virtual systems in the EL framework is worked in Chapter 7. Subsequently, the networked version of two different passivity-based tracking controllers in the literature are particular cases of the proposed technique.

Finally, conclusions, remarks and future perspectives of the results presented in this thesis are discussed in Chapter 8.

In Appendix A a self-contained prime on the differential geometry tools of nonlinear systems is presented. On the other hand, Appendix B presents a self-contained survey on the EL and pH approaches to mechanical mechanical systems.

Chapter 2

Preliminaries

”Once you get the physics right, the rest is mathematics.”

- Rudolf E. Kalman

In this chapter a self-contained survey of the differential approach to incremental stability by means of contraction analysis is presented. First two contraction analyses frameworks are introduced, the differential Finsler-Lyapunov framework and the logarithmic (matrix) measure. This is followed by the notion of differential passivity.
Next, the concept of virtual (control) systems is introduced as well as the relation of these systems with contraction analysis and differential passivity. Finally, their use in control design is discussed, which provides the set-up for virtual contraction based control.

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2.1 Contraction analysis and differential passivity

Most concepts are presented in local coordinates. However, in some cases it will be helpful to have a coordinate-free understanding of some of the objects. In this case, we refer to Appendix A where some geometry tools for nonlinear systems are presented. When it is clear from the context, arguments of some functions will be often left out.

2.1.1 Incremental stability

Let Σ_u be a nonlinear control system, affine in the input u, with state space manifold $\mathcal{X}$ of dimension N, which in local coordinates $(x_1, \ldots, x_N)$ is given by

$$\Sigma_u : \begin{cases} \dot{x} = f(x,t) + \sum_{i=1}^{n} g_i(x,t)u_i, \\ y = h(x,t), \end{cases} \tag{2.1}$$

where $x \in \mathcal{X}$ is the state, the input $u \in \mathcal{U} \subset \mathbb{R}^n$ is a measurable locally bounded function of time, and output $y \in \mathcal{Y} \subset \mathbb{R}^n$. The time-dependent vector fields[1] $f, g_i \in \mathfrak{X}^\infty(\mathcal{X})$ and the map $h \in C^\infty(\mathcal{X} \times \mathbb{R})$ are assumed to be smooth. The input space $\mathcal{U}$ and output space $\mathcal{Y}$ are open subsets of $\mathbb{R}^n$. The control system Σ_u in closed-loop with $u = \gamma(x,t)$ is

$$\Sigma : \begin{cases} \dot{x} = F(x,t) = f(x,t) + \sum_{i=1}^{n} g_i(x,t)\gamma_i(x,t), \\ y = h(x,t). \end{cases} \tag{2.2}$$

Solutions to Σ_u are given by trajectories $t \in [t_0, T] \mapsto x(t) = \psi^u_{t_0}(x_0, t)$ resulting from the initial condition $x_0 \in \mathcal{X}$, for a fixed input function $u : [t_0, T] \to \mathcal{U}$, with $\psi^{u_0}_{t_0}(x_0, t_0) = x_0$. Consider a forward invariant and connected open neighborhood C of $\mathcal{X}$ such that $\psi^u_{t_0}(t, x_0)$ is forward complete for every $x_0 \in C$, each function u and each t_0. Solutions to Σ are defined in a similar fashion and are denoted by $x(t) = \psi_{t_0}(x_0, t)$. By connectedness, any pair of points x_0, x_1 in C can be connected by a smooth curve $\gamma : (-\varepsilon, \varepsilon) \to C$, with $\gamma(-\epsilon) = x_0$ and $\gamma(\epsilon) = x_1$.

The general idea of incremental stability consists of comparing any pair of solutions of the system with respect to a distance. In this case we do not need to know the existence of an equilibrium or other reference solution in advance; as in Lyapunov's stability.

Definition 2.1 (Incremental stability (Forni and Sepulchre 2014))**.** *Let $C \subseteq \mathcal{X}$ be a forward invariant set, $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ be a continuous distance and consider the system Σ given by* (2.2)*. Then, system Σ is said to be*

- Incrementally stable *(Δ-S) on C (with respect to d) if there exists a function α of class $\mathcal{K}$*[2] *such that for each $x_1, x_2 \in C$, for each $t_0 \in \mathbb{R}_{\geq 0}$ and for all $t \geq t_0$,*

$$d(\psi_{t_0}(t, x_1), \psi_{t_0}(t, x_2)) \leq \alpha(d(x_1, x_2)). \tag{2.3}$$

[1]Vector fields f and g_i are maps $f, g_i : \mathcal{X} \times \mathbb{R} \to T\mathcal{X}$, with the properties $\pi \circ f = (id, 0)$ and $\pi \circ g_i = (id, 0)$, see Appendix A.2.

[2]Function α is said to be of class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$ (Khalil 1996).

- Incrementally asymptotically stable *(Δ-AS) on C if it is Δ-S and for all $x_1, x_2 \in C$, and for each $t_0 \in \mathbb{R}_{\geq 0}$,*

$$\lim_{t\to\infty} d(\psi_{t_0}(t, x_1), \psi_{t_0}(t, x_2)) = 0. \tag{2.4}$$

- Incrementally exponentially stable *(Δ-ES) on C if there exist a distance d, $k \geq 1$, and $\beta > 0$ such that for each $x_1, x_2 \in C$, fir each $t_0 \in \mathbb{R}_{\geq 0}$ and for all $t \geq t_0$,*

$$d(\psi_{t_0}(t, x_1), \psi_{t_0}(t, x_2)) \leq ke^{-\beta(t-t_0)} d(x_1, x_2). \tag{2.5}$$

If $C = \mathcal{X}$, then we say global Δ-S, Δ-AS and Δ-ES, respectively.

The above definitions are the incremental versions of the classical notions of stability, asymptotic stability and exponential stability (Khalil 1996).
A Lyapunov approach to incremental stability properties is presented in (Angeli 2002) where characterizations and applications are shown. However, in general, as in standard Lyapunov stability, finding a suitable Lyapunov function is difficult.

2.1.2 Differential Lyapunov theory and contraction analysis

Contraction analysis was introduced in (Lohmiller and Slotine 1998) as a differential alternative to study incremental stability on Euclidean or Riemannian state manifolds. More specifically, analyzing the dynamics of the system's state *first variation* (i.e., the linearization everywhere), one can conclude incremental stability via path integration. This idea was further generalized in (Forni and Sepulchre 2014) to systems on Finsler manifolds as follows. Let us introduce the following concepts.

Following (Crouch and van der Schaft 1987) and (Forni et al. 2013), the variational dynamics of systems Σ_u and Σ are defined as follows: Let $t \in [t_0, T] \mapsto x(t, s) = \psi_{t_0}(\gamma(s), t)$ be a s-parametrized family of state trajectories, from the initial condition $x(t_0, s) = \gamma(s)$, at time t_0. The corresponding family of input-output pair trajectories are $t \in [t_0, T] \mapsto u(t, s) = \varrho_{t_0}(t, s)$ and $t \in [t_0, T] \mapsto y(t, s) = \varsigma_{t_0}(t, s) = h(\psi_{t_0}(\gamma(s), t), t)$, for $s \in I = (-\varepsilon, \varepsilon)$. The differential $\frac{\partial}{\partial t}$ at a fixed s is the time derivative. Thus, $\psi_{t_0}(t, \gamma(s))$ satisfies in coordinates

$$\begin{aligned} \frac{\partial \psi_{t_0}}{\partial t}(t, \gamma(s)) &= f(\psi_{t_0}(t, \gamma(s)), t) + \sum_{i=1}^{n} g_i(\psi_{t_0}(t, \gamma(s)), t)\varrho_{i,t_0}(t, s), \\ \varsigma_{t_0}(t, s) &= h(\psi_{t_0}(t, \gamma(s)), t), \end{aligned} \tag{2.6}$$

for all $t \geq t_0$ and $s \in I$. On the other hand, the differential $\frac{\partial}{\partial s}$ at a fixed t is the infinitesimal variation with respect to s. Denote the nominal input-state-output trajectory by $u(t) = u(t,0)$, $x(t) = x(t,0)$ and $y(t) = y(t,0)$. Then, the variations of $(u(t), x(t), y(t))$ are

$$\delta u = \frac{\partial \varrho_{t_0}}{\partial s}(t,s)\Big|_{s=0}, \quad \delta x = \frac{\partial \psi_{t_0}}{\partial s}(t,\gamma(s))\Big|_{s=0} \quad \delta y = \frac{\partial \varsigma_{t_0}}{\partial s}(t,s)\Big|_{s=0}, \tag{2.7}$$

which are tangent to $\varrho_{t_0}(t,s)$, $\psi_{t_0}(t,\gamma(s))$, and $\varsigma_{t_0}(t,s)$ at s, respectively, i.e, $\delta u \in T_u\mathcal{U}$, $\delta x \in T_x\mathcal{X}$, and $\delta y \in T_y\mathcal{Y}$. The dynamics of the variational state $\delta x(t)$ is then

$$\begin{aligned}
\delta\dot{x}(t) &= \frac{\partial^2 \psi_{t_0}}{\partial t \partial s}(t,\gamma(0)) = \frac{\partial^2 \psi_{t_0}}{\partial s \partial t}(t,\gamma(0)),\\
&= \frac{\partial}{\partial s}\Big[f(\psi_{t_0}(t,\gamma(0)),t) + \sum_{i=1}^{n} g_i(\psi_{t_0}(t,\gamma(0)),t)\varrho_{i,t_0}(t,0)\Big],\\
&= \frac{\partial f}{\partial x}(\psi_{t_0}(t,\gamma(0)),t)\frac{\partial \psi_{t_0}}{\partial s}(t,\gamma(0))\\
&\quad + \sum_{i=1}^{n}\frac{\partial g_i}{\partial x}(\psi_{t_0}(t,\gamma(0)),t)\frac{\partial \psi_{t_0}}{\partial s}(t,\gamma(0))\varrho_{i,t_0}(t,0)\\
&\quad + \sum_{i=1}^{n} g_i(\psi_{t_0}(t,\gamma(0)),t)\frac{\partial \varrho_{i,t_0}}{\partial s}(t,0).
\end{aligned} \tag{2.8}$$

Thus, the *variational system/dynamics* of Σ_u in (2.1) along the the trajectory $(u,x,y)(t)$ is the time-varying system given and denoted by

$$\delta\Sigma_u : \begin{cases} \delta\dot{x} = \frac{\partial f}{\partial x}(x,t)\delta x + \sum_{i=1}^{n} u_i \frac{\partial g_i}{\partial x}(x,t)\delta x + \sum_{i=1}^{n} g_i \delta u_i, \\ \delta y = \frac{\partial h}{\partial x}(x,t)\delta x. \end{cases} \tag{2.9}$$

Definition 2.2 ((Crouch and van der Schaft 1987))**.** *The* prolonged control system Σ_u^δ *associated to the control system* Σ_u *in* (2.1) *corresponds to consider the original system* Σ_u *and its variational system* $\delta\Sigma_u$*, that is, the system described by*

$$\Sigma_u^\delta : \begin{cases} \dot{x} = f(x,t) + \sum_{i=1}^{n} g_i(x,t)u_i, \\ y = h(x,t), \\ \delta\dot{x} = \frac{\partial f}{\partial x}\delta x + \sum_{i=1}^{n} u_i \frac{\partial g_i}{\partial x}\delta x + \sum_{i=1}^{n} g_i \delta u_i, \\ \delta y = \frac{\partial h}{\partial x}(x,t)\delta x. \end{cases} \tag{2.10}$$

with $(u,\delta u) \in T\mathcal{U}$*,* $(x,\delta x) \in T\mathcal{X}$*, and* $(y,\delta y) \in T\mathcal{Y}$*. The prolonged system* Σ^δ *of the* *closed system* Σ *in* (2.2) *is similarly defined as*

$$\Sigma^\delta : \begin{cases} \dot{x} = F(x,t), \\ y = h(x,t), \\ \delta\dot{x} = \frac{\partial F}{\partial x}(x,t)\delta x, \\ \delta y = \frac{\partial h}{\partial x}(x,t)\delta x. \end{cases} \tag{2.11}$$

Definition 2.3 (Finsler structure)**.** *The function* $\mathfrak{F} : T\mathcal{X} \times \mathbb{R} \to \mathbb{R}_{\geq 0}$ *defines a Finsler structure if it satisfies the following conditions:*

- $\mathfrak{F}$ *is a uniform* C^1 *function on* $T\mathcal{X} \times \mathbb{R}$ *for* $\delta x \neq 0$*;*
- $\mathfrak{F}(x, \delta x, t) > 0$ *for each* $(x, \delta x) \in T\mathcal{X}$ *uniformly in t such that* $\delta x \neq 0$*;*
- $\mathfrak{F}(x, \lambda\delta x, t) = \lambda\mathfrak{F}(x, \delta x, t)$ *for each* $\lambda \geq 0$ *and for each* $(x, \delta x) \in T\mathcal{X}$ *uniformly in t, such that* $\delta x \neq 0$ *(homogeneity);*
- $\mathfrak{F}(x, \delta x_1 + \delta x_2, t) \leq \mathfrak{F}(x, \delta x_1, t) + \mathfrak{F}(x, \delta x_2, t)$*, for each* $(x, \delta x_1), (x, \delta x_2) \in T\mathcal{X}$ *uniformly in t (convexity).*

Positiveness, homogeneity, and strict convexity of $\mathfrak{F}$ guarantee that $\mathfrak{F}(x, \cdot, t)$ is a Minkowski norm on each tangent space. The length of any curve $\gamma(s)$ induced by $\mathfrak{F}$ is independent of orientation-preserving re-parametrizations.

Definition 2.4 (Differential Lyapunov function)**.** *Let* $\mathfrak{F}(x, \delta x, t)$ *be a Finsler structure. A function* $V : T\mathcal{X} \times \mathbb{R} \to \mathbb{R}_{>0}$ *is a candidate* differential Lyapunov function *adapted to* $\mathfrak{F}$ *if it satisfies*

$$c_1\mathfrak{F}(x, \delta x, t)^p \leq V(x, \delta x, t) \leq c_2\mathfrak{F}(x, \delta x, t)^p, \tag{2.12}$$

for some constants $c_1, c_2 \in \mathbb{R}_{>0}$*, with p a positive integer.*

The relation between a candidate differential Lyapunov function and the Finsler structure in (2.12) is the *key property* for incremental stability analysis. That is, a uniformly well-defined distance on $\mathcal{X} \times \mathbb{R}$ via integration as defined below.

Definition 2.5. *Consider a candidate differential Lyapunov function on* $\mathcal{X}$ *and the associated Finsler structure* $\mathfrak{F}$*. For any subset* $C \subseteq \mathcal{X}$ *and any* $x_1, x_2 \in C$*, let* $\Gamma(x_1, x_2)$ *be the collection of piecewise* C^1 *curves* $\gamma : I \to \mathcal{X}$ *connecting* $\gamma(0) = x_1$ *and* $\gamma(1) = x_2$*.* The Finsler distance $d : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_{\geq 0}$ *induced by* $\mathfrak{F}$ *is defined by*

$$d(x_1, x_2) := \inf_{\gamma \in \Gamma(x_1, x_2)} \int_\gamma \mathfrak{F}\left(\gamma(s), \frac{\partial \gamma}{\partial s}(s), t\right) ds. \tag{2.13}$$

The following result gives a sufficient condition for incremental stability in terms of differential Lyapunov functions as in (Forni and Sepulchre 2014). This can be seen as the differential/Finsler-Lyapunov version of the direct or second Lyapunov method.

Theorem 2.6 (Differential Lyapunov method)**.** *Consider the prolonged system Σ^δ, a connected and forward invariant set $C \subseteq \mathcal{X}$, and a function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Let V be a candidate differential Lyapunov function satisfying*

$$\dot{V}(x, \delta x, t) \leq -\alpha(V(x, \delta x, t)) \tag{2.14}$$

for each $(x, \delta x, t) \in T\mathcal{X} \times \mathbb{R}$. Then, system Σ in (2.2) is

- *Δ-S on C uniformly in t, if $\alpha(s) = 0$ for each $s \geq 0$;*
- *Δ-AS on C uniformly in t, if α is a $\mathcal{K}$ function;*
- *Δ-ES on C uniformly in t, if $\alpha(s) = \beta s, \forall s > 0$.*

In the following figure a geometric interpretation of this is shown

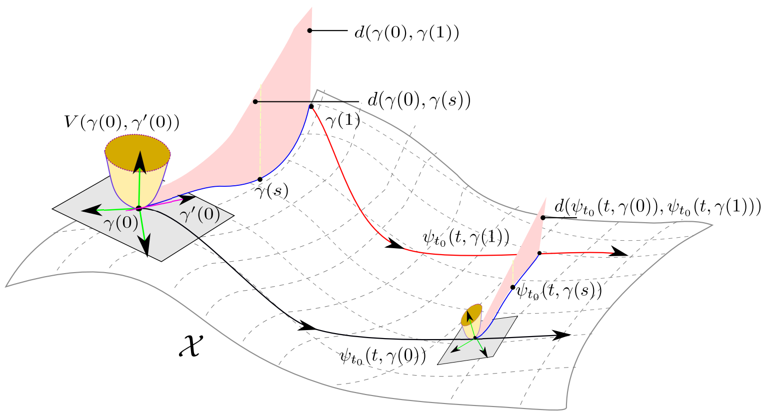

Figure 2.1: Geometric interpretation of Theorem 2.6.

We are ready to give a definition of contraction in terms of differential Lyapunov functions as follows, (Forni and Sepulchre 2014) and (Sanfelice and Praly 2015):

Definition 2.7 (Contraction/Non-expansiveness)**.** *We say that Σ* contracts *(respectively* does not expand*) V in C if* (2.14) *is satisfied for a function α of class $\mathcal{K}$ (resp. $\alpha(s) = 0$ for all $s \geq 0$). C is* the contraction region *(resp.* nonexpanding region*).*

Contraction analysis via Riemannian metrics

Consider the Finsler structure

$$\mathfrak{F}(x, \delta x, t) = \sqrt{\frac{1}{2}\delta x^\top \Pi(x, t)\delta x}, \tag{2.15}$$

with $\Pi(x, t)$ a Riemannian metric tensor, possibly depending on t. Then, a corresponding candidate differential Lyapunov function is given by

$$V(x, \delta x, t) = \mathfrak{F}(x, \delta x, t)^2 = \frac{1}{2}\delta x^\top \Pi(x, t)\delta x. \tag{2.16}$$

In this case, condition (2.14) amounts to the *generalized contraction analysis* condition as in (Lohmiller and Slotine 1998), given by

$$\dot{\Pi}(x, t) + \frac{\partial F^\top}{\partial x}(x, t)\Pi(x, t) + \Pi(x, t)\frac{\partial F}{\partial x}(x, t) \leq -2\beta(x, t)\Pi(x, t). \tag{2.17}$$

for some $\beta(x, t) > 0$. Under hypotheses of Theorem 2.6, if $C \subseteq \mathcal{X}$ is also a compact set, then system (2.2) satisfies the following definition (Rüffer et al. 2013):

Definition 2.8 (Convergent systems (Pavlov and van de Wouw 2017))**.** *System* Σ *in* (2.2) *with initial condition* $x_0 \in C$ *is called* uniformly convergent *on* C *if*

1. *there is a unique solution* $\overline{x}(t)$ *that is defined and bounded on* C*, for all* $t \in \mathbb{R}_{\geq 0}$*,*
2. $\overline{x}(t)$ *is uniformly asymptotically stable on* C*.*

If $\overline{x}(t)$ *is uniformly exponentially stable, then system* (2.2) *is called* uniformly exponentially convergent.

Remark 2.9 (Demidovich condition (Pavlov et al. 2004))**.** *Suppose there exist constant positive definite matrices* $\Pi = \Pi^\top$ *and* $Q = Q^\top$ *such that*

$$\Pi\frac{\partial F}{\partial x}(x, t) + \frac{\partial F^\top}{\partial x}(x, t)\Pi \leq -Q \tag{2.18}$$

Then system (2.2) *is uniformly exponentially convergent. Clearly, condition* (2.18) *reduces to* (2.17) *with* $\Pi(x, t) = \Pi$ *constant. Hence,* (2.14) *is a generalization of* (2.18).

Contraction analysis via Logarithmic matrix measures

A different approach to contraction analysis is taken in (di Bernardo et al. 2009, Russo et al. 2010, Sontag 2010) in terms of the so-called *matrix measure*.

Definition 2.10 (Matrix measure/Logarithmic norm)**.** *Given a vector norm $\|\cdot\|$ on the Euclidean space $\mathbb{R}^n$, with its induced matrix norm $\|A\|$, the associated* matrix measure *μ is defined as the directional derivative of the matrix norm in the direction of A and evaluated at the identity matrix I_n, that is,*

$$\mu(A) := \lim_{h \searrow 0} \frac{1}{h} \left(\|I_n + hA\| - 1 \right). \tag{2.19}$$

The limit exists and the convergence is monotonic (Aminzarey and Sontag 2014). Some vector norms and their induced matrix measures are shown in Table 2.1.

Vector norm $\|\cdot\|_p$	Induced matrix measure $\mu_p(A)$
$\|\delta x\|_1 = \sum_{i=1}^n \|\delta x_i\|$	$\mu_1(A) = \max_j \left(a_{jj} + \sum_{i \neq j} \|a_{ij}\| \right)$
$\|\delta x\|_2 = \left(\sum_{i=1}^n \|\delta x_i\|^2 \right)^{1/2}$	$\mu_2(A) = \max_{\lambda \in \mathrm{spec} \frac{1}{2}(A + A^\top)} \lambda$
$\|\delta x\|_\infty = \max_{1 \leq i \leq n} \|\delta x_i\|$	$\mu_\infty(A) = \max_i \left(a_{ii} + \sum_{i \neq j} \|a_{ij}\| \right)$

Table 2.1: Matrix measures for a matrix $A \in \mathbb{R}^{n \times n}$.

Definition 2.11. *Given a norm $\|\cdot\|_p$, the system Σ in* (2.2)*, or the time-dependent vector field $F(x,t)$, is called* infinitesimally contracting *with respect to this norm on a set $C \subseteq \mathcal{X}$ if there exist some norm in $T_x\mathcal{X}$, with associated induced matrix measure μ_p, such that, for some constant 2β (the contraction rate), it holds that*

$$\mu_p \left(\frac{\partial F}{\partial x}(x,t) \right) \leq -2\beta, \quad \textit{for all } x \in C, \textit{ and all } t \geq 0. \tag{2.20}$$

If this is satisfied, any pair of solutions of (2.2) *converge to each other with rate β.*

Suppose that $\Pi(x,t)$ in (2.17) is written as $\Pi(x,t) = \Theta^\top(x,t)\Theta(x,t)$, see (di Bernardo et al. 2009). Then, the Riemannian contraction condition in (2.17) is equivalent to the matrix measure contraction condition given by

$$\mu(\overline{J}(x,t)) \leq -2\beta, \tag{2.21}$$

where the *generalized Jacobian* ((Lohmiller and Slotine 1998)) is given by

$$\overline{J}(x,t) = \left[\dot{\Theta}(x,t)F(x,t) + \Theta(x,t)\frac{\partial F}{\partial x}(x,t) \right] \Theta^{-1}(x,t), \tag{2.22}$$

as shown in (di Bernardo et al. 2009, Forni and Sepulchre 2014, Coogan 2017).

2.1.3 Differential passivity

In analogy to the standard dissipativity theory (Willems 1972), the differential Lyapunov framework is extended to systems with inputs as follows (Forni and Sepulchre 2013, Forni et al. 2013, van der Schaft 2013).

Definition 2.12 (Differential passivity). *Consider a nonlinear control system Σ_u as in* (2.1) *together with its prolonged system Σ_u^δ given by* (2.10). *Then, Σ_u is* differentially passive *if the prolonged system Σ_u^δ is dissipative with respect to the supply rate $\delta y^\top \delta u$, i.e., if there exist a* differential storage function *function $W : T\mathcal{X} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ satisfying*

$$\frac{dW}{dt}(x, \delta x, t) \leq \delta y^\top \delta u, \tag{2.23}$$

for all $x, \delta x, u, \delta u$, and for all t. Furthermore, system (2.1) *is called* differentially lossless *if* (2.23) *holds with equality.*

If additionally is required the differential storage function to be a differential Lyapunov function, then differential passivity implies contraction when the variational input is $\delta u = 0$. For further details we refer to (van der Schaft 2013) and (Forni et al. 2013).

The following lemma characterizes the structure of a class of control systems which are differentially passive.

Lemma 2.13 ((Reyes-Báez, van der Schaft and Jayawardhana 2019)). *Consider the control system Σ_u in* (2.1) *together with its prolonged system Σ_u^δ in* (2.10). *Suppose there exists a transformation $\delta\tilde{x} = \Theta(x, t)\delta x$ such that the variational dynamics in* (2.9) *takes the form*

$$\delta\tilde{\Sigma}_u : \begin{cases} \delta\dot{\tilde{x}} = [\Xi(\tilde{x}, t) - \Upsilon(\tilde{x}, t)]\, \Pi(\tilde{x}, t)\delta\tilde{x} + \Psi(\tilde{x}, t)\delta u, \\ \delta\tilde{y} = \Psi^\top(\tilde{x}, t)\Pi(\tilde{x}, t)\delta\tilde{x}, \end{cases} \tag{2.24}$$

where $\Pi(\tilde{x}, t) > 0_N$ is a Riemannian metric tensor, $\Xi(\tilde{x}, t) = -\Xi^\top(\tilde{x}, t)$ and $\Upsilon(\tilde{x}, t)$ are rectangular matrices. If the condition

$$\delta\tilde{x}^\top \left[\dot{\Pi}(\tilde{x}, t) - \Pi(\tilde{x}, t)(\Upsilon(\tilde{x}, t) + \Upsilon^\top(\tilde{x}, t))\Pi(\tilde{x}, t)\right]\delta\tilde{x} \leq -\alpha(W(\tilde{x}, \delta\tilde{x}, t)), \tag{2.25}$$

holds for all $(\tilde{x}, \delta\tilde{x}) \in T\mathcal{X}$, and all t, with α of class $\mathcal{K}$, then Σ_u is differentially passive from δu to $\delta\tilde{y}$ with respect to the differential storage function given by

$$W(\tilde{x}, \delta\tilde{x}, t) = \frac{1}{2}\delta\tilde{x}^\top \Pi(\tilde{x}, t)\delta\tilde{x}. \tag{2.26}$$

Proof It is straightforward to check that the time derivative of (2.26) is

$$\begin{aligned}\dot{W} &= \delta\tilde{x}^\top \Pi(\tilde{x},t)\left[\Xi(\tilde{x},t) - \Upsilon(\tilde{x},t)\right]\Pi(\tilde{x},t)\delta\tilde{x} \\ &\quad + \frac{1}{2}\delta\tilde{x}^\top \dot{\Pi}(\tilde{x},t)\delta\tilde{x} + \delta\tilde{x}^\top \Pi(\tilde{x},t)\Psi(\tilde{x},t)\delta u, \\ &= \frac{1}{2}\delta\tilde{x}\left[\dot{\Pi}(\tilde{x},t) - \Pi(\tilde{x},t)(\Upsilon(\tilde{x},t) + \Upsilon^\top(\tilde{x},t))\Pi(\tilde{x},t)\right]\delta\tilde{x} + \delta y^\top \delta u, \\ &\leq -\alpha(W(\tilde{x},\delta\tilde{x})) + \delta y^\top \delta\omega \leq \delta y^\top \delta u. \qquad \blacksquare\end{aligned} \tag{2.27}$$

The passivity theorem of negative feedback interconnection of two passive systems resulting in a passive closed-loop system can be extended to differential passivity as follows. Consider two differentially passive nonlinear systems Σ_{u_i}, with states $x_i \in \mathcal{X}_i$, inputs $u_i \in \mathcal{Y}_i$, outputs $u_i \in \mathcal{U}$ and differential storage functions W_i, with $i \in \{1,2\}$. The standard feedback interconnection is

$$u_1 = -y_2 + e_1, \quad u_2 = y_1 + e_2, \tag{2.28}$$

where e_1, e_2 denote external outputs. The equations (2.28) imply that the variational quantities $\delta u_1, \delta u_2, \delta y_1, \delta y_2, \delta e_1, \delta e_2$ satisfy

$$\delta u_1 = -\delta y_2 + \delta e_1, \quad \delta u_2 = \delta y_1 + \delta e_2. \tag{2.29}$$

It follows that

$$\delta u_1^\top \delta y_1 + \delta u_2^\top \delta y_2 = \delta e_1^\top \delta y_1 + \delta e_2^\top \delta y_2, \tag{2.30}$$

and thus, the closed-loop system arising from the feedback interconnection in (2.29) of Σ_{u_1} and Σ_{u_2} is a differentially passive system with supply rate $\delta e_1^\top \delta y_1 + \delta e_2^\top \delta y_2$ and storage function $W = W_1 + W_2$, as it is shown by (van der Schaft 2013).

2.2 Virtual contraction analysis and control

We first introduce the notion of virtual (control) systems, followed by its relation with contraction analysis and differential passivity. Finally, we address the methodology of virtual contraction based control (v-CBC) for nonlinear affine systems.

2.2.1 Virtual systems

In the following definition the different notions of virtual system introduced in (Lohmiller and Slotine 1998, Wang and Slotine 2005, Jouffroy and Fossen 2010, Forni and Sepulchre 2014, van der Schaft 2017) are unified and generalized.

Definition 2.14 (Virtual system). *Consider systems Σ_u and Σ, given by* (2.1) *and* (2.2)*, respectively. Suppose that $C_v \subseteq \mathcal{X}$ and $C_x \subseteq \mathcal{X}$ are connected and forward invariant for* (2.1) *and* (2.2)*, respectively. A* virtual control system *associated to Σ_u is defined as*

$$\Sigma_u^v : \begin{cases} \dot{x}_v = \Gamma_v(x_v, x, u_v, t), \\ y_v = h_v(x_v, x, t), \qquad \forall t \geq t_0, \end{cases} \tag{2.31}$$

with state $x_v \in \mathcal{X}$ and parametrized by $x \in \mathcal{X}$, where $h_v : C_v \times C_x \times \mathbb{R}_{\geq 0} \to \mathcal{Y}$ and $\Gamma_v : C_v \times C_x \times \mathcal{U} \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ are such that

$$\Gamma(x, x, u, t) = f(x, t) + \sum_{i=1}^{n} g_i(x, t)u_i, \quad \text{and} \quad h_v(x, x, t) = h(x, t), \forall u, \forall t \geq t_0. \tag{2.32}$$

Similarly, a virtual system *associated to Σ is defined as*

$$\Sigma^v : \begin{cases} \dot{x}_v = \Phi_v(x_v, x, t), \\ y_v = h_v(x_v, x, t), \end{cases} \tag{2.33}$$

with state $x_v \in C_v$ and parametrized by $x \in C_x$, where $\Phi_v : C_v \times C_x \times \mathbb{R}_{\geq 0} \to T\mathcal{X}$ and $h_v : C_v \times C_x \times \mathbb{R}_{\geq 0} \to \mathcal{Y}$ satisfy

$$\Phi_v(x, x, t) = F(x, t) \quad \text{and} \quad h_v(x, x, t) = h(x, t), \quad \forall x, \forall t \geq t_0. \tag{2.34}$$

It follows that any solution $x(t) = \psi_{t_0}(t, x_o)$ of the *original control system* Σ_u in (2.1), starting at $x_0 \in C_x$ for a certain input u, generates the solution $x_v(t) = \psi_{t_0}(t, x_0)$ to the virtual system Σ_u^v in (2.31), starting at $x_{v0} = x_0 \in C_v$ with $u_v = u$, for all $t > t_0$. In a similar manner for the original system Σ in (2.2), any solution $x(t) = \psi_{t_0}(t, x_o)$ starting at $x_0 \in C_x$, generates the solution $x_v(t) = \psi_{t_0}(t, x_o)$ to the closed virtual system Σ^v in (2.33), starting at $x_{v0} = x_0 \in C_v$, for all $t > t_0$. However, *not* every virtual system's solution $x_v(t)$ corresponds to an original system's solution. Thus, *for* any *trajectory $x(t)$ of the original system, we may consider* (2.31) *(respectively* (2.33)*) as a time-varying (or parameter varying) system with state x_v.*

Example 2.15 ((Wang and Slotine 2005, Jouffroy and Fossen 2010)). *The previous definition is illustrated in the following academic example. Consider the control system*

$$\begin{aligned} \dot{x} &= -D(x)x + u, \\ y &= x. \end{aligned} \tag{2.35}$$

with $x \in \mathbb{R}$, $D(x) > 0$, and input $u \in \mathbb{R}$. It is straightforward to verify that the system

$$\begin{aligned} \dot{x}_v &= -D(x)x_v + u_v, \\ y_v &= x_v, \end{aligned} \tag{2.36}$$

with state $x_v \in \mathbb{R}$, input $u_v \in \mathbb{R}$ and parametrized by x, is a virtual system for (2.35). *Indeed, whenever $u_v = u$ and $x_{v0} = x_0$, the system* (2.36) *produces the same input-state-output behavior of the original system in* (2.35).

2.2.2 Virtual contraction analysis

A generalization of contraction analysis was first introduced in (Wang and Slotine 2005) and revisited in (Jouffroy and Fossen 2010, Forni and Sepulchre 2014), to study the convergence between solutions of two or more (possibly different) systems. This concept, referred to as *virtual contraction analysis*, is based on the contraction behavior of a virtual (control) system as shown bellow:

Theorem 2.16 (Virtual contraction)**.** *Consider systems Σ and Σ^v given by* (2.2) *and* (2.33)*, respectively. Let $C_v \subseteq \mathcal{X}$ and $C_x \subseteq \mathcal{X}$ be two connected and forward invariant sets. Suppose that Σ^v is uniformly contracting with respect to x_v. Then, for any initial conditions $x_0 \in C_x$ and $x_{v0} \in C_v$, each solution to Σ^v converges asymptotically to the solution of Σ.*

Proof: Let $t \in [0, T] \mapsto x_v(t) = \xi_{t_0}(t, x_{z0})$ be the solution to system Σ^v starting from $x_{z0} \in C_v$, at time t_0. With the solution to system Σ given by $x(t) = \psi_{t_0}(t, x_0)$, the virtual system Σ^v can be rewritten as $\dot{x}_v = \Phi_v(x_v, \psi_{t_0}(t, x_0), t)$. Since Σ^v is contracting (for all x), then $\lim_{t\to\infty} d(\xi_{t_0}(t, x_{v10}), \xi_{t_0}(t, x_{v20})) = 0$, with $x_{v10}, x_{v20} \in C_v$. In particular, whenever $x_{v10} = x_0$, we have that $x_{v1} = \psi_{t_0}(t, x_0)$, with $x_0 \in C_x$ (due to $\Phi(x, x, t) = F(x, t)$). Hence, for every $x_{v20} \in C_v$, $\lim_{t\to\infty} d(\psi_{t_0}(t, x_0), \xi_{t_0}(t, x_{v20})) = 0$. ■

If the conditions of Theorem 2.16 hold, then the original system Σ is said to be *virtually contractive*. Notice that, if C_v is compact and Σ^v is contractive, then Σ^v is also convergent (see Definition 2.8) and $\overline{x}_v(t) = x(t)$ is the steady state solution.

If the virtual control system Σ_u^v is differentially passive, then the original control system Σ_u is said to be *virtually differentially passive*. In this case, the steady-state solution is driven by the input and is denoted by $\overline{x}_v^{u_v}(t) = x^u(t)$. This key property can be used for v-CBC design, as it will be shown in the next subsection.

Example 2.17 (Continued)**.** *Consider the prolonged system of* (2.36) *is given by*

$$\begin{aligned} \dot{x}_v &= -D(x)x_v + u_v, \\ y_v &= x_v, \\ \delta\dot{x}_v &= -D(x)\delta x_v + \delta u_v, \\ \delta y_v &= \delta x_v, \end{aligned} \tag{2.37}$$

together with the differential Lyapunov function (see (2.16)*)*

$$V(x_v, \delta x_v, x) = \frac{1}{2}\delta x_v^2. \tag{2.38}$$

The time derivative along the solutions of the prolonged system (2.37) *is*

$$\dot{V}(x_v, \delta x_v, x) = -\delta x_v D(x)\delta x_v + \delta x_v \delta u_v \leq \delta y_v u_v, \tag{2.39}$$

which shows that the original system (2.35) *is virtually differentially passive. Furthermore, if* $u_v = 0$ *then the original system* (2.35) *is virtually contractive.*

2.2.3 Virtual contraction based control (v-CBC)

From a control point of view the usual task is to render a specific solution of the system exponentially/asymptotically stable, rather than the stronger contractive behavior of all system's solutions. In this regard, as an alternative to the existing techniques in the literature, we propose a technique based on the concept of virtual contraction in order to solve the set-point regulation or trajectory tracking problems.

Roughly speaking, the control objective is to design a scheme such that a distance between a desired steady-state solution $x_d(t)$ and the original system's solution x shrinks as a consequence of contractive behavior of the closed-loop virtual system.

The proposed design methodology is divided in three main steps:

1. Consider a virtual control system Σ_u^v as in (2.31) for system Σ_u in (2.1).
2. Design a state feedback $u_v = \zeta(x_v, x, t) + \omega_v$ for Σ_u^v such that the closed-loop virtual system is contracting and has a desired solution $x_d(t)$ when the external input $\omega = 0$.
3. Define the controller for the original control system Σ_u as $u = \zeta(x, x, t)$.

If we are able to design a controller with the above steps, then according to Theorem 2.16, all the solutions of the closed-loop virtual system will converge to the closed-loop original system solution starting at x_0, that is, $\overline{x}(t) = x_d(t) \to x(t)$ as $t \to \infty$. This solves the set-point or trajectory tracking problem for the original system Σ_u.

Example 2.18 (Continued)**.** *The goal is to design a tracking controller for the original system* (2.35) *via the v-CBC method for a given desired trajectory $x_d(t)$. For the step 1, consider the virtual control system* (2.36)*. Later, for the step 2, take the feedback law*

$$u_v = -K_p(x_v - x_d) + \omega_v, \quad K_p > 0. \tag{2.40}$$

for the virtual system (2.36)*. Then, the resulting closed-loop prolonged virtual system is given by*

$$\begin{aligned} \dot{x}_v &= -D(x)x_v - K_p(x_v - x_d) + \omega_v, \\ y_v &= x_v, \\ \delta\dot{x}_v &= -\Big[D(x) + K_p\Big]\delta x_v + \delta\omega_v, \\ \delta y_v &= \delta x_v. \end{aligned} \tag{2.41}$$

In order to show that the closed-loop virtual system is differentially passive, consider as differential storage function to (2.38)*. The time derivative along system* (2.41) *is*

$$\dot{V}(x_v, \delta x_v, x) = -\delta x_v \Big[D(x) + K_p\Big]\delta x_v + \delta x_v \delta\omega_v \leq \delta y_v \omega_v, \tag{2.42}$$

which completes the proof. It follows that for $\omega = 0$ the closed-loop virtual system is contractive and that $x_v = x_d(t)$ is a particular solution or the closed loop system in (2.41)*. Hence, all the solutions of the virtual closed-loop virtual system in* (2.41) *converge to $x_d(t)$.*

Finally, for the step 3, close the loop of the original system (2.35) *with the scheme*

$$u = -K_p(x - x_d) + \omega \tag{2.43}$$

yielding the closed-loop system

$$\begin{aligned} \dot{x} &= -D(x)x - K_p(x - x_d) + \omega, \\ y &= x. \end{aligned} \tag{2.44}$$

The conclusion follows by the virtual contraction Theorem 2.16 since (2.41) *is the prolonged system of the original system* (2.44)*. Therefore, x converges to $x_d(t)$.*

2.2.4 Trajectory tracking via v-CBC

One of the central topics of this thesis is the control synthesis for solving the trajectory tracking problem in nonlinear mechanical systems. In the following lines, the aforementioned problem is stated and a solution to his via the v-CBC method is proposed.

Trajectory tracking problem: Given a desired smooth trajectory $x_d(t)$ for system Σ_u, design the control law u such that $x(t)$ converges asymptotically/exponentially to the desired trajectory $x_d(t)$.

Proposed solution: Using the v-CBC method in Section 2.2.3, design a control scheme with the following structure:

$$\zeta(x_v, x, t) := u_v^{ff}(x_v, x, t) + u_v^{fb}(x_v, x, t) \tag{2.45}$$

where the *feedforward-like* term u_v^{ff} ensures that the closed-loop virtual system has the desired trajectory $x_d(t)$ as steady-state solution, and the *feedback* action u_v^{fb} enforces the closed-loop virtual system to be contracting.

Chapter 3

Energy-based virtual mechanical systems

> ”If you want to find the secrets of the universe, think in terms of energy, frequency and vibration.”
>
> - Nikola Tesla

In this chapter a class of virtual control systems associated to mechanical systems in the Euler-Lagrange and port-Hamiltonian energy-based frameworks are introduced. We show how these virtual systems inherit some properties of the original ones, for instance energy conservation. Furthermore, these virtual systems exhibit some coordinate free properties. Finally, we elaborate on the application of such virtual systems in control design. The main results in this chapter are partially reported in the conference paper (Reyes-Báez, Borja, van der Schaft and Jayawardhana 2019).

3.1 Virtual systems in the Euler-Lagrange framework

Consider the Euler-Lagrange equations[1] (EL) given by

$$\begin{aligned} \dot{q} &= v, \\ M(q)\dot{v} + C(q,v)v + g(q) &= B(q)\tau, \end{aligned} \tag{3.1}$$

where $\dot{q} \in \mathcal{Q}$ is the generalized position on the configuration space $\mathcal{Q}$, $v = \dot{q} \in T_q\mathcal{Q}$ is the velocity, $M(q)$ is the inertia matrix which is positive-definite and bounded; $C(q,v)$ is the Coriolis and centrifugal forces matrix, and $g(q)$ is the vector of gravitational forces. The covector $B(q)\tau$, with inputs $\tau \in \mathcal{U}$, represents the vector of external forces. Matrix $B(q)$ indicates how the action of the inputs τ influences the system. If rank $B(q) = m < n$ then we say that system (3.1) is *underactuated*.

It is well known that the EL equations (3.1) exhibit several important dynamics properties; see (Ortega et al. 2013) and references therein for further details. Among those

[1] See Appendix B.1 for a self-contained survey on Euler-Lagrange equations and the notation used in this chapter. The underlying Riemannian geometry interpretation of some properties is also presented.

properties, the skew-symmetry of the matrix $N(q,\dot{q}) := \dot{M}(q) - 2C(q,\dot{q})$ receives special attention since it is closely related to the energy conservation of the EL system (3.1). To see this consider the total (co-)energy

$$\mathcal{E}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top M(q)\dot{q} + P(q), \tag{3.2}$$

where $P(q)$ is the potential energy. Then, the time derivative of (3.2) given by

$$\dot{\mathcal{E}}(q,\dot{q}) = \dot{q}^\top B(q)\tau + \frac{1}{2}\dot{q}^\top N(q,\dot{q})\dot{q} = \dot{q}^\top B(q)\tau, \tag{3.3}$$

shows that the increase of energy is equal to the supplied energy. In the dissipativity theory setting (Willems 1972, van der Schaft 2017), the system (3.1) is called *lossless*. We refer to (Ortega et al. 2013) for the passivity approach to EL systems.

From a Riemannian point of view, the skew-symmetry of matrix $N(q,\dot{q})$ is a clear expression in *local coordinates* of the torsion-free property and compatibility condition of the *Levi-Civita affine connection* $\overset{M}{\nabla}$ with the metric $M\langle v,v\rangle := v^\top M(q)v$ (see Appendix B.1.2 for a detailed explanation). These imply that the skew-symmetric matrix $N(q,\dot{q})$ can be equivalently rewritten as follows:

$$\dot{q}_v^\top \left[\dot{M}(q) - 2C(q,\dot{q})\right]\dot{q}_v = 0 \iff L_{\dot{q}}(M\langle\dot{q}_v,\dot{q}_v\rangle) - 2M\langle\overset{M}{\nabla}_{\dot{q}}\dot{q}_v,\dot{q}_v\rangle = 0, \quad \forall \dot{q}_v \in T_qQ. \tag{3.4}$$

where $L_{\dot{q}}(M\langle\dot{q}_v,\dot{q}_v\rangle)$ is the Lie derivative[2] of the metric $M\langle v,v\rangle$ along the velocity $\dot{q}$, and $\overset{M}{\nabla}_{\dot{q}}\dot{q}_v$ is the *covariant derivative* of the tangent vector $\dot{q}_v = Y(q) \in T_qQ$ along the velocity $\dot{q} = X(q)$, whose expression in local coordinates is given by (see details in Section B.1.1)

$$\overset{M}{\nabla}_{X(q)}Y(q) = \frac{\partial Y}{\partial q}(q)X(q) + M^{-1}(q)C(q,X(q))Y(q). \tag{3.5}$$

Remark 3.1. *The energy conservation condition* (3.3) *requires the identity* (3.4) *to hold only for $\dot{q}_v = \dot{q}$, rather than for every tangent vectors $\dot{q}_v \in T_qQ$.*

Notice that condition (3.4) implies that the forces induced by $N(q,\dot{q})$ defined as

$$F_N(q,\dot{q}) := N(q,\dot{q})\dot{q} \quad \text{and} \quad F_{N_v}(q,\dot{q},\dot{q}_v) := N(q,\dot{q})\dot{q}_v \tag{3.6}$$

are *workless*[3]. This means that their corresponding power is given by $\dot{q}^\top F(q,\dot{q}) = 0$ and $\dot{q}_v^\top F_{N_v}(q,\dot{q},\dot{q}_v) = 0$, respectively. However, it should be noted that $F_{N_v}(q,\dot{q},\dot{q}_v)$ may not

[2]See Appendix A.2 a definition of Lie derivative.

[3] See Section B.1.3 for further details on workless forces.

define a *true force* since the tangent vector $\dot{q}_v \in T_q Q$ need not to be equal to the velocity of q. In case that $\dot{q}_v = \dot{q}$ holds, then the identity

$$F_{N_v}(q, \dot{q}, \dot{q}) = F_N(q, \dot{q}) \tag{3.7}$$

is satisfied. Such $F_{N_v}(q, \dot{q}, \dot{q}_v)$ is said to be a *virtual force*.

Exploiting the above introduced notion of virtual forces, in the next proposition we define a virtual control system associated to the (original) EL system (3.1).

Proposition 3.2. *Consider the EL system* (3.1)*. Consider also the system defined by*

$$\begin{aligned} \dot{q}_v &= v_v, \\ M(q)\dot{v}_v + C(q, v)v_v + g_v(q_v) &= B(q)\tau_v, \\ y_v &= B^\top(q)v_v, \end{aligned} \tag{3.8}$$

in the state $(q_v, v_v) \in Q \times \mathbb{R}^n$ *and parametrized by the vector* $(q, v = \dot{q}) \in TQ$*, where* $g_v(q_v)$ *is such that* $g_v(q) = g(q)$*,* $B(q)\tau_v \in T^*Q$ *is a covector with inputs* τ_v*, and where* y_v *is an output. Then, system* (3.8) *defines a virtual control system for the original system* (3.1).

Proof: Consider system (3.1) in state space form (2.1) with $x = (q, v)$ and vector fields

$$f(x,t) = \begin{bmatrix} v \\ M^{-1}(q)\left(-C(q, v)v - g(q)\right) \end{bmatrix}, \quad \sum_{i=1}^{n} g_i(x,t)u_i = \begin{bmatrix} 0 \\ M^{-1}(q)B(q) \end{bmatrix}\tau, \tag{3.9}$$

input $u = \tau$, and let $h(x, t) = B^\top(q)v$ be the output function. Similarly, consider the state space form (2.31) of the virtual system (3.8) with state $x_v = (q_v, v_v)$, vector field

$$\Gamma_v(q_v, v_v, q, v, \tau_v, t) = \begin{bmatrix} v_v \\ M^{-1}(q)\left(B(q)\tau_v - C(q, v)v_v - g_v(q_v)\right) \end{bmatrix}, \tag{3.10}$$

and output function $h_v(q_v, v_v, q, v, \tau_v, t) = B^\top(q)v_v$. By taking $(q_v, v_v) = (q, v)$ and $\tau_v = \tau$, respectively, system (3.8) satisfies conditions (2.32) of Definition 2.14. Therefore, (3.8) is a virtual system for the original EL system (3.1). ■

Note that the functions $C(q, \dot{q})v_v$ and $M(q)\dot{v}_v$ are virtual forces associated to the Coriolis force $C(q, \dot{q})v$ and the inertial force $M(q)\dot{v}$ of system (3.1), respectively.

Remark 3.3. *The virtual system* (3.8) *can be seen as the second-order version of the one introduced in (van der Schaft 2017, Definition 4.6.2).*

3.1.1 Losslessness preserving property

Remarkably, not only the EL system (3.1) is lossless (i.e., preserves the energy) with respect to the output $y = B^\top(q)v$, but also the virtual system (3.8) turns out to be lossless with respect to the output $y_v = B^\top(q)v_v$, *for every time-function* $(q(\cdot), v(\cdot) = \dot{q}(\cdot))$. As we will see in the following proposition, the skew-symmetry of $N(q, v)$ is crucially used in the computations of the proof. To this end, consider the storage function of the virtual system (3.8) as the function of (q_v, v_v) given by

$$\mathcal{E}_v(q_v, v_v, q, v) := \frac{1}{2} v_v^\top M(q) v_v + P_v(q_v), \tag{3.11}$$

parametrized by (q, v), where $P_v(q_v)$ is such that $g_v(q_v) = \partial P_v / \partial q_v(q_v)$.

Proposition 3.4. *For any curve* $(q(\cdot), v(\cdot) = \dot{q}(\cdot))$ *the virtual system* (3.8) *with input* τ_v *and output* y_v *is lossless, with* (q, v)*-parametrized storage function* (3.11).

Proof: Consider the storage function (3.11). Then, the energy balance reads as

$$\begin{aligned}
\dot{\mathcal{E}}_v(q_v, v_v, q, v) &= \frac{1}{2} v_v^\top \dot{M}(q) v_v + v_v^\top M(q) \dot{v}_v + \frac{\partial P_v^\top}{\partial q_v}(q_v) \dot{q}_v, \\
&= \frac{1}{2} v_v^\top \dot{M}(q) v_v + v_v^\top \left(-C(q, v) v_v - g_v(q_v) + B(q) \tau_v\right) + \frac{\partial P_v^\top}{\partial q_v}(q_v) v_v, \\
&= v_v^\top B(q) \tau_v = y_v^\top \tau_v.
\end{aligned} \tag{3.12}$$

■

Proposition 3.4 can be easily extended to the case when the original EL system (3.1) contains dissipative forces. In particular, if the dissipation is generated by a Rayleigh function $R(\dot{q})$ satisfying $\dot{q}^\top \frac{\partial R}{\partial \dot{q}}(\dot{q}) \geq 0$, then the virtual system (3.8) is passive with input-output pair (τ_v, y_v) and storage function (3.11). Indeed,

$$\dot{\mathcal{E}}_v(q_v, v_v, q, v) \leq -\dot{q}_v^\top \frac{\partial R}{\partial \dot{q}}(\dot{q}_v) \dot{q}_v + y_v^\top \tau_v. \tag{3.13}$$

Moreover, Proposition 3.4 give us the opportunity of extending the standard losslessness (or passivity) preserving interconnection properties of the original EL system (3.1) to the virtual system (3.8). In Chapter 7, these interconnection properties will be exploited to design control protocols for the group coordination problem.

3.1.2 Coordinate-free description

Consider the local coordinates expression of the covariant derivative $\overset{M}{\nabla}_{X(q)}Y(q)$ in (3.5) of the tangent vector $\dot{q}_v = Y(q) = X(q)$ along $\dot{q} = X(q)$, namely

$$\begin{aligned} \overset{M}{\nabla}_{\dot{q}(t)}\dot{q}(t) &= \frac{\partial X}{\partial q}(q)X(q) + M^{-1}(q)C(q, X(q))X(q), \\ &= \frac{\mathrm{d}X}{\mathrm{d}t}(q(t)) + M^{-1}(q(t))C(q, X(q(t)))X(q(t)), \\ &= \ddot{q}(t) + M^{-1}(q(t))C(q(t), \dot{q}(t))\dot{q}(t). \end{aligned} \tag{3.14}$$

Hence, the EL system (3.1) can be expressed in a coordinate-free manner in the context of Riemannian geometry as follows:

$$\begin{aligned} \dot{q}(t) &= X(q), \\ \overset{M}{\nabla}_{X(q(t))}X(q(t)) + \mathrm{grad}(P(q(t))) &= M^{-1}(q(t))B(q(t))\tau, \end{aligned} \tag{3.15}$$

where $\mathrm{grad}(P(q)) \in T_qQ$ is the gradient of the potential energy function $P(q)$ which in local coordinates is given by $\mathrm{grad}(P(q)) = M^{-1}(q)\frac{\partial P}{\partial q}$. For the external input term $M^{-1}(q(t))B(q(t))\tau$ we note that from a geometric point of view, the force $B(q)\tau$ is an element of the cotangent space T_q^*Q. In this case, $M^{-1}(q)$ defines a map from the cotangent space and to tangent space; this yields $M^{-1}(q(t))B(q(t))\tau \in T_qQ$. Expression (3.15) means that whenever the EL system (3.1) is at an equilibrium state in the absence of external forces τ, then the dynamics reduces to the *geodesic equation*

$$\overset{M}{\nabla}_{\dot{q}(t)}\dot{q}(t) = 0. \tag{3.16}$$

From the notion of virtual forces previously introduced together with the above observations and the losslessness property preservation in Proposition 3.4, we conclude that a coordinate-free description of the virtual system (3.8) is given by

$$\begin{aligned} \dot{q}_v(t) &= Y(q_v), \\ \overset{M}{\nabla}_{X(q)}Y(q_v(t)) + \mathrm{grad}(P(q(t))) &= M^{-1}(q(t))B(q(t))\tau_v. \end{aligned} \tag{3.17}$$

Therefore the state $\dot{q}_v(t)$ at any moment t is an element of $T_{q(t)}Q$. By definition of affine connection (see Appendix B.1.1), $\overset{M}{\nabla}_{\dot{q}(t)}Y(q_v(t))$ depends on the vector field X only through its value $X(q(t))$. Hence at every time t the covariant derivative $\overset{M}{\nabla}_{\dot{q}(t)}Y(q_v(t))$ depends on the curve $q(\cdot)$ only through the value $\dot{q} \in T_{q(t)}Q$.

We highlight that we can take any curve $q(t)$ in Q with velocity vector field $\dot{q}(t) = X(q(t))$, and consider the virtual dynamics (3.17) of *any* vector field $\dot{q}_v = Y(q_v(t))$ *along* this curve $q(t)$, i.e., $\dot{q}_v \in T_{q(t)}Q$. If we take $\dot{q}_v = \dot{q}$, then (3.17) reduces to (3.15).

3.1.3 Applications in control design

The virtual control system (3.8) has been used in nonlinear control design mainly in two different ways: as a *target closed-loop system* or as an *auxiliary system* on which the controller convergence-analysis/design is performed.

As a target closed-loop system, (3.8) first appeared in the context of tracking control of the fully-actuated robots with dynamic model given by (3.1) (Slotine and Li 1987, Ortega et al. 2013). To this end, consider the control scheme defined as[4]

$$\tau := M(q)\dot{v}_r + C(q,v)v_r + \frac{\partial P}{\partial q}(q) + \tau_v, \tag{3.18}$$

where $v_r := \dot{q}_d - \Lambda(q - q_d)$ is an artificial velocity reference, q_d a desired curve on Q, $K \in \mathbb{R}^{n\times n}$ is a positive definite matrix gain and τ_e is an external input. Without loss of generality, take $B(q) = I_n$. Then, system (3.1) in closed-loop with (3.18) yields

$$M(q)\dot{v} + C(q,v)v + \frac{\partial P}{\partial q}(q) = M(q)\dot{v}_r + C(q,v)v_r + \frac{\partial P}{\partial q}(q) + v, \tag{3.19}$$

which in the velocity "error" coordinate $v_v := v - v_r$ takes the form

$$\begin{aligned} \dot{q}_v &= v_v, \\ M(q)\dot{v}_v + C(q,v)v_v &= \tau_v, \end{aligned} \tag{3.20}$$

that is, (3.8) with $P_v(q_v) = 0$. By Proposition 3.4, $(q_v, v_v) = (0_n, 0_n)$ is a globally stable[5] equilibrium point of the closed-loop system (3.20).

On the other hand, considering the inner feedback loop $\tau_v = -K(v_v - v) + \overline{\tau}_v$ in the virtual system (3.8) the resulting system is contractive; see (Jouffroy and Fossen 2010)

[4]From (3.11) and (3.18), the reference velocity v_r satisfies $v_r^\top N(q,v)v_r = 0$. Hence $v_r \in T_qQ$.

[5]Asymptotic stability is ensured by adding an extra feedback loop via v. For instance, defining $v = -Kv_v + \tau_e$, with $K \in \mathbb{R}^{n\times n}$, the closed-loop system is *strictly output passive* with input-output pair (τ_e, v_v), implying that the equilibrium $(q_v, v_v) = (0_n, 0_n)$ is globally asymptotically stable for $\tau_e = 0_n$; see (van der Schaft 2017, Example 4.6.5) for a proof. On the other hand, if $v = -K\mathrm{sign}(v_v)$, the closed-loop system has a *sliding mode* (Sira-Ramírez 2015) and the origin is reached in *finite-time*.

In both cases, the "error" (called *sliding variable*) $v_v = v - v_r = \tilde{v} - \Lambda\tilde{q}$, with $\tilde{v} = v - v_d$ and $\tilde{q} = q - q_d$, defines an invariant and attractive subset of Q given by $v_v = 0$, called the *sliding manifold*. The dynamics in $v_v = 0$ is given by $\dot{\tilde{q}} = -\Lambda\tilde{q}$, and it is referred as the *ideal* sliding dynamics.

for details. Since both the controller (3.18) and the original EL system (3.8) define two particular solutions, that is, $v_v = v$ and $v_v = v_r$, respectively. It follows that v converges to v_r exponentially. See Example 2.17.

Similarly, but from a design perspective, in (Manchester et al. 2018) the standard stabilization/tracking problem to a desired solution $q_d(\cdot)$ is solved for the original EL system (3.1). This is done by designing a scheme directly for the virtual system (3.8) that drives it to be contractive in the closed-loop. Then, as in the previous case, the closed-loop original system and the controller are particular solutions of the closed-loop contractive system. This lies in the v-CBC method in Section 2.2.3. See Example 2.18.

3.2 Virtual systems in the port-Hamiltonian framework

As an alternative to the described Euler-Lagrange framework for mechanical systems, the (port-)Hamiltonian formulation[6] can be adopted. In particular, in this work we consider the class of input-state-output port-Hamiltonian (pH) systems defined as follows (van der Schaft and Jeltsema 2014).

Definition 3.5. *An* input-state-output port-Hamiltonian system *with N-dimensional state space manifold $\mathcal{X}$, input and output spaces $\mathcal{U} = \mathcal{Y} \subset \mathbb{R}^m$, and Hamiltonian function $H : \mathcal{X} \to \mathbb{R}$, is given as*

$$\begin{aligned} \dot{x} &= [J(x) - R(x)] \frac{\partial H}{\partial x}(x) + g(x)u \\ y &= g^\top(x) \frac{\partial H}{\partial x}(x), \end{aligned} \tag{3.21}$$

where $g(x)$ is a $N \times m$ input matrix, $J(x)$ and $R(x)$ are the interconnection and dissipation $N \times N$ matrices which satisfy $J(x) = -J^\top(x)$ and $R(x) = R^\top(x) \geq 0$.

For a mechanical system with configuration manifold $\mathcal{Q}$, we have that the state space is given by $\mathcal{X} = T^*\mathcal{Q}$ with natural coordinates $x = (q, p) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$, and the Hamiltonian function corresponds to the total energy defined by

$$H(q, p) = \frac{1}{2} p^\top M^{-1}(q) p + P(q), \tag{3.22}$$

[6]A survey of the Hamiltonian formulation of mechanical systems is presented in Appendix B.2.

where $p := M(q)\dot{q}$ is the generalized momentum. Then, system (3.21) takes the form

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_n & I_n \\ -I_n & -D(q,p) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0_n \\ B(q) \end{bmatrix} \tau, \\ y &= B^\top(q) \frac{\partial H}{\partial p}(q,p), \end{aligned} \tag{3.23}$$

with $\tau = u \in \mathbb{R}^m$, interconnection, dissipation and input matrices given by

$$J(x) = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}, \quad R(x) = \begin{bmatrix} 0_n & 0_n \\ 0_n & D(q,p) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0_n \\ B(q) \end{bmatrix}, \tag{3.24}$$

respectively. The $n \times n$ matrix $D(q,p) = D^\top(q,p) \geq 0_n$ is a dissipation term; and I_n and 0_n are the $n \times n$ identity and zero matrices, respectively. The input force matrix $B(q)$ has rank $m \leq n$. If $n = m$ we say that the system (3.23) is *fully-actuated*; if $m < n$, we say (3.23) is *underactuated*. Moreover, if $m > n$ then system (3.23) is *overactuated*.

Similar to the EL framework, the energy balance for system (3.23) is given by the time derivative of the Hamiltonian (3.22) along the system's trajectories, i.e.,

$$\dot{H}(q,p) = \frac{\partial H^\top}{\partial q}(q,p)\dot{q} + \frac{\partial H^\top}{\partial p}(q,p)\dot{p} = -\frac{\partial H^\top}{\partial p}(q,p) D(q,p) \frac{\partial H}{\partial p}(q,p) + \tau \leq y^\top \tau. \tag{3.25}$$

It follows that the map $\tau \mapsto y$ is *passive* (van der Schaft 2017) with storage function (3.22). Furthermore, the system is *lossless* if $D(q,p) = 0_n$.
System in (3.23) can be equivalently rewritten as follows (Reyes-Báez, van der Schaft and Jayawardhana 2019)

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(q,p) + D(q,p)) \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0 \\ B(q) \end{bmatrix} \tau, \\ y_E &= \begin{bmatrix} 0_n & B^\top(q) \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}, \end{aligned} \tag{3.26}$$

where $E(q,p) := S_H(q,p) - \frac{1}{2}\dot{M}(q)$, with $S_H(q,p)$ a skew-symmetric matrix as in (B.72), and the output $y_E = y$. The main characteristic of this alternative form is that the forces associated to the inertia matrix of (3.23), i.e. $\frac{\partial}{\partial q}(\frac{1}{2}p^\top M^{-1}(q)p)$, are decoupled from the force $\frac{\partial H}{\partial q}(q,p)$ by means of the identity

$$\frac{\partial}{\partial q}\left(\frac{1}{2}p^\top M^{-1}(q)p\right) = E(q,p)M^{-1}(q)p. \tag{3.27}$$

Notice that this is possible without any change of coordinates or feedback, contrary to the literature; e.g. (Venkatraman et al. 2010) and (Romero, Ortega and Sarras 2015). Moreover, it can be shown that the force $E(q,p)M^{-1}(q)p$ is the Hamiltonian counterpart[7] of the force $C(q,\dot{q})\dot{q}$ in the Lagrangian framework. A detailed derivation of (3.26)-(3.27) and their properties are presented in Appendix B.2.3.

As expected, the input-output pair (u, y_E) defines a passive map with the same storage function (3.22). Indeed, the time derivative of $H(q,p)$ along the trajectories of the alternative representation (3.26) is given by

$$\begin{aligned}\dot{H} &= \begin{bmatrix} \frac{\partial P}{\partial q} + \frac{\partial}{\partial q}\left(\frac{1}{2}p^\top M^{-1}p\right) \\ \frac{\partial H}{\partial p} \end{bmatrix}^\top \left(\begin{bmatrix} 0_n & I_n \\ -I_n & -(E+D) \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} + \begin{bmatrix} 0 \\ B \end{bmatrix} \tau \right) \\ &= \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}^\top \begin{bmatrix} 0_n & I_n \\ -I_n & -2S_H(q,p) \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} - \frac{\partial H^\top}{\partial p} D(q,p) \frac{\partial H}{\partial p} + y_E^\top \tau \leq y_E^\top \tau. \end{aligned} \tag{3.28}$$

The last inequality follows from the positive definiteness of $D(q)$ and from

$$\begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}^\top \underbrace{\begin{bmatrix} 0_n & I_n \\ -I_n & -E \end{bmatrix}}_{:=\mathcal{F}(q,p)} \begin{bmatrix} \frac{\partial P}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} = \begin{bmatrix} \frac{\partial P}{\partial q} \\ \frac{\partial H}{\partial p} \\ \frac{\partial}{\partial q}\left(\frac{1}{2}p^\top M^{-1}p\right) \end{bmatrix}^\top \underbrace{\begin{bmatrix} 0_n & I_n & 0_n \\ -I_n & -E(q,p) & 0_n \\ 0_n & 0_n & I_n \end{bmatrix}}_{:=\overline{\mathcal{F}}(q,p)} \begin{bmatrix} \frac{\partial P}{\partial q} \\ \frac{\partial H}{\partial p} \\ M^{-1}(q)p \end{bmatrix} = 0. \tag{3.29}$$

Let $\mathbb{F}(q,p) : C^\infty(T^*Q) \to \mathbb{R}^{2n}$ be the map[8] defined in coordinates as

$$\mathbb{F}(q,p)[H(q,p)] := \mathcal{F}(q,p) \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}, \qquad H(q,p) \in C^\infty(T^*Q), \tag{3.30}$$

with $H(q,p)$ given in (3.22). From the energy balance (3.29), it follows that

$$\frac{\partial H^\top}{\partial x}(q,p)\mathbb{F}(q,p)[H(q,p)] = 0. \tag{3.31}$$

This suggests that the quantity $\mathbb{F}(q,p)[H(q,p)]$ can be interpreted as the Hamiltonian counterpart of the workless force $F_N(q,\dot{q})$ in (3.6).

Similarly, consider a smooth function $H_v(q_v, p_v, q) \in C^\infty(T^*Q \times Q)$ of the form

$$H_v(q_v, p_v, q) = \frac{1}{2} p_v^\top M^{-1}(q) p_v + P_v(q_v), \tag{3.32}$$

[7]Similar result were presented in (Sarras et al. 2012) and (Stadlmayr and Schlacher 2008). We refer the interested reader to Remark B.7 in Appendix B.2.3 for further details.

[8]$C^\infty(T^*Q)$ is the set of smooth scalar functions $H : T^*Q \to \mathbb{R}$. See Appendix A for further details.

and parametrized by $q(\cdot)$ from (3.26), where function $P_v(q_v)$ is such that $P_v(q) = P(q)$. Consider also the map $\overline{\mathbb{F}}(q,p) : C^\infty(T^*Q \times Q) \to \mathbb{R}^{3n}$ defined as

$$\overline{\mathbb{F}}(q,p)[H_v(q_v,p_v,q)] := \overline{\mathcal{F}}(q,p)\begin{bmatrix} \frac{\partial P_v}{\partial q_v}(q_v) \\ \frac{\partial H_v}{\partial p_v}(q_v,p_v,q) \\ M^{-1}(q)p \end{bmatrix}. \tag{3.33}$$

By construction, it straightforward to verify that

$$\frac{\partial H_v^\top}{\partial \overline{x}_v}(q_v,p_v,q)\overline{\mathbb{F}}(q,p)[H(q_v,p_v,q)] = 0, \tag{3.34}$$

where $\overline{x}_v = (x_v, q)$, $x_v = (q_v, p_v)$ and q is the solution to $\dot{q} = M^{-1}(q)p$. This in turn implies that whenever $(q_v,p_v) = (q,p)$ it follows that

$$\frac{\partial H_v^\top}{\partial \overline{x}_v}(q,p,q)\overline{\mathbb{F}}(q,p)[H(q,p,q)] = \frac{\partial H^\top}{\partial x}(q,p)\mathbb{F}(q,p)[H(q,p)]. \tag{3.35}$$

Hence, $\overline{\mathbb{F}}(q,p)[H_v(q_v,p_v,q)]$ can be understood as the Hamiltonian counterpart of the workless virtual force $F_{N_v}(q,\dot{q},q_v)$ in (3.6).

From the above observations in the following proposition a virtual system associated to the mechanical pH system (3.23) is constructed.

Proposition 3.6. *Consider system* (3.23) *and its alternative coordinate formulation* (3.26). *Consider also the input-state-output system given by*

$$\begin{aligned} \begin{bmatrix} \dot{q}_v \\ \dot{p}_v \end{bmatrix} &= \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(q,p)+D(q,p)) \end{bmatrix}\begin{bmatrix} \frac{\partial H_v}{\partial q_v}(q_v,p_v,q) \\ \frac{\partial H_v}{\partial p_v}(q_v,p_v,q) \end{bmatrix} + \begin{bmatrix} 0_n \\ B(q) \end{bmatrix}\tau_v \\ y_v &= \begin{bmatrix} 0_n & B^\top(q) \end{bmatrix}\begin{bmatrix} \frac{\partial H_v}{\partial q_v}(q_v,p_v,q) \\ \frac{\partial H_v}{\partial p_v}(q_v,p_v,q) \end{bmatrix}, \end{aligned} \tag{3.36}$$

in the state $(q_v,p_v) \in Q \times \mathbb{R}^n$ and parametrized by $(q,p) \in T^*Q$, where $H_v(q_v,p_v,q)$ is given as in (3.32), *$B(q)\tau_v \in T^*Q$ is a covector with inputs τ_v, and y_v is an output. Then, system* (3.36) *defines a virtual control system for the original system* (3.23).

Proof: System (3.23) satisfies the affine form in (2.1) with state $x = (q,p)$; and drift, respectively, input vector fields given by

$$f(x,t) = \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(q,p)+D(q,p)) \end{bmatrix}\begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}, \quad \sum_{i=1}^{n} g_i(x,t)u_i = \begin{bmatrix} 0 \\ B(q) \end{bmatrix}\tau. \tag{3.37}$$

In this case, the input is $u = \tau$ and the output function is defined as $h(x,t) = B^\top(q)M^{-1}(q)p$. Similarly, system (3.36) is of the form as (2.31) with

$$\Gamma_v(q_v, p_v, q, p, \tau_v, t) = \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(q,p) + D(q,p)) \end{bmatrix} \begin{bmatrix} \frac{\partial H_v}{\partial q_v}(q_v, p_v, q) \\ \frac{\partial H_v}{\partial p_v}(q_v, p_v, q) \end{bmatrix} + \begin{bmatrix} 0_n \\ B(q) \end{bmatrix} u_v, \quad (3.38)$$

and output function $h_v(q_v, p_v, q, p, \tau_v, t) = B^\top(q)\frac{\partial H_v}{\partial p_v}(q_v, p_v, q)$. By taking $(q_v, p_v) = (q, p)$ and $\tau_v = \tau$, system (3.8) satisfies conditions (2.32) of Definition 2.14. It follows that system (3.8) is a virtual system for the alternative form (3.26). Moreover, by virtue of the identity (3.27), system (3.36) is also a virtual system for (3.23) ■

3.2.1 Structure preserving property

Similar to its EL counterpart, not only the pH system (3.23) (respectively the alternative form (3.26)) is passive with input-output pair (τ, y) (respectively (τ, y_E)) and storage function (3.22), but also the virtual system (3.37) with input-output pair (τ_v, y_v) and storage function given by (3.32), for every time-functions $(q(\cdot), p(\cdot))$. In the following proposition it is shown that the passivity preserving property relies in the "workless" property of the map $\overline{\mathbb{F}}(q,p)$ in (3.34).

Proposition 3.7. *For any curve* $(q(\cdot), p(\cdot))$ *the virtual system* (3.37) *with input* τ_v *and output* y_v *is passive, with q-parametrized storage function* (3.32)*.*

Proof: Consider the storage function (3.32). Then, the energy balance reads as

$$\begin{aligned} \dot{H}_v(q_v, p_v, q) &= \frac{\partial H_v^\top}{\partial \overline{x}_v}(q_v, p_v, q)\overline{\mathbb{F}}(q,p)[H_v(q_v, p_v, q)] \\ &\quad - p^\top M^{-1}(q)D(q,p)M^{-1}(q)p_v + y_v^\top \tau_v, \\ &\leq y_v^\top \tau_v. \end{aligned} \quad (3.39)$$

■

In the absence of dissipation ($D(q,p) = 0_n$), the system is lossless.

From (3.32) we see that $\frac{\partial H_v}{\partial q_v}(q_v, p_v, q) = \frac{\partial P}{\partial q_v}(q_v)$ holds. Hence, the virtual system (3.36) can be rewritten as follows

$$\begin{aligned} \dot{x}_v &= [\mathcal{J}_v(q,p) - \mathcal{R}_v(q,p)] \frac{\partial H_v}{\partial x_v}(q_v, p_v, q) + g(x)u_v, \\ y_v &= g^\top(x)\frac{\partial H_v}{\partial x_v}(q_v, p_v, q), \end{aligned} \quad (3.40)$$

where $x_v = (q_v, p_v)$, matrix $g(x)$ is given as in (3.24), and the matrices

$$\mathcal{J}_v(q,p) = \begin{bmatrix} 0_n & I_n \\ -I_n & -S_H(q,p) \end{bmatrix}, \quad \text{and} \quad \mathcal{R}_v(q,p) := \begin{bmatrix} 0_n & 0_n \\ 0_n & (D(q,p) - \frac{1}{2}\dot{M}(q)) \end{bmatrix}, \tag{3.41}$$

satisfy $\mathcal{J}_v(q,p) = -\mathcal{J}_v^\top(q,p)$ and $\mathcal{R}_v(q,p) = \mathcal{R}_v^\top(q,p)$, respectively. Notice that the virtual system (3.40)-(3.41) has the form of (3.21). However, matrix $\mathcal{R}_v(q,p)$ may not be positive semi-definite. Due to this we say that (3.40) is a *mechanical pH-like system*.

Consider the variational virtual system of (3.40), with respect to x_v (see (2.9))

$$\begin{aligned} \delta\dot{x}_v &= [\mathcal{J}_v(q,p) - \mathcal{R}_v(q,p)] \frac{\partial^2 H_v}{\partial x_v^2}(q_v, p_v, q)\delta x_v + g(x)\delta u_v, \\ \delta y_v &= g^\top(x)\frac{\partial^2 H_v}{\partial x_v^2}(q_v, p_v, q)\delta x_v. \end{aligned} \tag{3.42}$$

Notice that the variational system is of the form (2.24) with $\Xi(x_v, x) = J_v(q,p)$, $\Upsilon(x_v, x) = R_v(q,p)$, and metric $\Pi(x_v, x) = \frac{\partial^2 H_v}{\partial x_v^2}(q_v, p_v, q)$, system (3.42). Moreover, sufficient conditions under which (3.40) is differentially passive can be given.

Corollary 3.8. *Assume that* $\frac{\partial^2 H_v}{\partial x_v^2}(q_v, p_v, q) > 0$ *and that the following condition holds*

$$\delta q_v^\top \frac{d}{dt}\left(\frac{\partial^2 P_v}{\partial q_v^2}(q_v)\right)\delta q_v \leq \alpha\left(\delta q_v^\top \frac{\partial^2 P_v}{\partial q_v^2}(q_v)\delta q_v\right). \tag{3.43}$$

Then, system (3.40) *is* differentially passive *with differential storage function*

$$W(x_v, \delta x_v, x) = \frac{1}{2}\delta x_v^\top \frac{\partial^2 H_v}{\partial x_v^2}(q_v, p_v, q)\delta x_v. \tag{3.44}$$

Proof: Consider the prolonged system composed by(3.40) and (3.42). Under assumption (3.43), system (3.40) satisfies hypothesis in Lemma 2.13. This completes the proof.

■

Indeed, condition (3.43) restricts the class of systems that are differentially passive in the open-loop. In the following this is relaxed 1by feedback control.

3.2.2 Coordinate-free interpretation

In order to explain some of the underlying geometric properties of the virtual system (3.40), we make use of the *(almost) Poisson manifolds*[9] setting of Hamiltonian systems. For a detailed explanation of this we refer the interested reader to Appendix B.2.1.

The interconnection skew-symmetric matrix $J(x)$ in (3.24) is the local representation in natural coordinates $x = (q, p)$ of the so-called *structure matrix* associated to the canonical (almost) *Poisson bracket* $\{\cdot, \cdot\}$, whose coordinate expression is given by

$$\{F, H\}(x) = \sum_{i,j=1}^{r} J_{ij}(x)\frac{\partial F}{\partial x_i}(x)\frac{\partial H}{\partial x_j}(x), \qquad F, H \in C^\infty(T^*Q). \tag{3.45}$$

Moreover, the structure matrix $J(x)$ satisfies the integrability condition[10]

$$\sum_{l=1}^{r}\left(J_{lj}\frac{\partial J_{ik}}{\partial x_l} + J_{li}\frac{\partial J_{kj}}{\partial x_l} + J_{lk}\frac{\partial J_{ji}}{\partial x_l}\right) = 0, \quad i, j, k \in \{1, \dots, r = 2n\}. \tag{3.46}$$

The above implies that the map $J(x) : T_x^*(T^*Q) \to T_x(T^*Q)$ is the matrix representation of the *bundle map* $J^\sharp : T^*(T^*Q) \to T(T^*Q)$, called the *sharp map*. This in turn implies that the Poisson bracket induces a skew-symmetric and nondegenerate bilinear form given by $J : T^*(T^*Q) \times T^*(T^*Q) \to \mathbb{R}$ such that[11]

$$J(dF, dH) = \langle dF, J^\sharp(x)dH\rangle. \tag{3.47}$$

Assuming that there is no dissipation (i.e., $D(q) = 0_n$), the pH system (3.23) can be expressed in a coordinate-free manner as follows:

$$\begin{aligned} \dot{x} &= J^\sharp(x)dH(x) + \sum_{i=1}^{m} g_i(x)u_i, \\ y_i &= \langle dH(x), g_i(x)\rangle, \quad i \in \{1, \dots m\} \end{aligned} \tag{3.48}$$

where $g_i \in \mathfrak{X}^\infty(T^*Q)$ are input vector fields. Then, the energy balance (3.28) in terms of the bilinear form (3.47) and the intrinsic system (3.48) as

$$\dot{H} = J(dH, dH) + \langle dH, \sum_{i=1}^{m} g_i u_i\rangle = \sum_{i=1}^{m}\langle dH, g_i u_i\rangle = \sum_{i=1}^{m} y_i u_i = y^\top u. \tag{3.49}$$

[9]The differential geometric tools used in this section are explained in Appendix A.
[10]This condition means that the Poisson bracket satisfies the Jacobi identity (B.48c).
[11]Or in local coordinated $\langle dF, J^\sharp(x)dH\rangle = \frac{\partial F^\top}{\partial x}(x)J(x)\frac{\partial H}{\partial x}(x)$.

It is clear that $J(dH, dH) = 0$. Furthermore, (3.34) is also a local expression of this.

In the case of the virtual pH-like system (3.40), rewritten as

$$\begin{aligned}\dot{x}_v &= \mathcal{J}_v(q,p)\frac{\partial H_v}{\partial x_v}(q_v,p_v,q) - \mathcal{R}_v(q,p)\frac{\partial H_v}{\partial x_v}(q_v,p_v,q) + \sum_{i=1}^{m} g_i(x)u_{v,i},\\ y_v &= g^\top(x)\frac{\partial H_v}{\partial x_v}(q_v,p_v,q),\end{aligned} \tag{3.50}$$

we have that $\mathcal{J}_v(q,p)$ given in (3.41) is skew-symmetric and parametrized by (q,p). Similar to (3.45), $\mathcal{J}_v(q,p)$ can be understood as the structure matrix of the bracket whose expression in coordinates $x_v = (q_v, p_v)$ is given by

$$\{F_v, H_v\}(x_v, q, p) = \sum_{i,j=1}^{r} \mathcal{J}_{v,ij}(q,p)\frac{\partial F}{\partial x_{v,i}}(x_v,q)\frac{\partial H}{\partial x_{v,j}}(x_v,q), \tag{3.51}$$

with $F_v, H_v \in C^\infty(T^*Q \times Q)$. However, the structure matrix $\mathcal{J}_v(q,p)$ does not satisfy its corresponding integrability condition (3.46). In this case, the bracket (3.51) is called an almost Poisson bracket, (see Appendix B.2.2). Pointwise at a fixed (q,p), the map $\mathcal{J}_v(q,p) : T^*_x(T^*Q \times Q) \to T_x(T^*Q \times Q)$ can be interpreted as the matrix representation of the *bundle map* $\mathcal{J}_v^\sharp(q,p) : T^*(T^*Q \times Q) \to T(T^*Q \times Q)$. Furthermore, it induces a skew-symmetric and nondegenerate bilinear form $\mathcal{J}_v : T^*(T^*Q\times Q)\times T^*(T^*Q\times Q) \to \mathbb{R}$

$$\mathcal{J}_v(dF_v, dH_v) = \langle dF_v, \mathcal{J}_v^\sharp(q,p)dH_v\rangle. \tag{3.52}$$

3.2.3 Applications to control design

Similar to the EL framework, virtual systems have been used in control design techniques for pH systems. Specifically, the structure preserving control techniques propose virtual systems as target behaviors. For instance, partial linearizion (Venkatraman et al. 2010), interconnection and Damping assignment passivity-Based Control (IDA-PBC) (Ortega et al. 2002), control by interconnection (CbI) method (Ortega et al. 2008), among others. In these methods, the virtual systems are constructed after an intermediate change of coordinates[12] that lets them to rewrite (3.23) as a system whose inertia matrix is constant.

On the other hand, in order to construct the virtual system (3.36), the aforementioned change of coordinates is *not necessary*. In fact, by exploiting the energy-conservation

[12]The main reason for this change of coordinates is that a Hamiltonian counterpart of the skew-symmetric matrix $N(q,v) = \dot{M} - 2C(q,v)$ was not available in the literature.

identity in (3.27), the gyroscopic terms are decoupled from $\frac{\partial H}{\partial q}\left(\frac{1}{2}p^\top M^{-1}(q)p\right)$. The virtual system (3.36) has only been used as a target system in (Reyes-Báez et al. 2017); where the port-Hamiltonian counterpart of the Slotine-Li controller in (3.18) is developed. It remains open to consider the virtual system (3.36) is taken as target behavior in the IDA or CbI methods, as well as the adaptive pH counterpart of (3.18) .

A different point of attack has been taken in (Reyes-Báez et al. 2017) and (Reyes-Báez et al. 2018) for solving the (tracking) control problem of the pH system (3.23) using the v-CBC method in Section 2.2.3.

Chapter 4

Virtual contraction based control of mechanical port-Hamiltonian systems

" The central conception of all modern theory in physics is the Hamiltonian. If you wish to apply modern theory to any particular problem, you must start with putting the problem in Hamiltonian form."

- Erwin Schrodinger

In this chapter the (trajectory) control problem of fully-actuated nonlinear mechanical systems in the port-Hamiltonian framework is solved via the virtual contraction-based control (v-CBC) method described in Section 2.2.3. This results in a large family of (exponentially) asymptotically stable (tracking) controllers with different structural in the closed-loop. The performance of some control schemes of the aforementioned family is evaluated experimentally on a planar robot of two degrees of freedom by Quanser. The results have been reported in (Reyes-Báez et al. 2017) and (Reyes-Báez, van der Schaft and Jayawardhana 2019).

4.1 Control design procedure via v-CBC

In this section the vCDC methodology described in Section 2.2.3 will be used to solve the tracking control design problem for system (3.23) under the following assumption.

Assumption 4.1. *For the mechanical system* (3.23) *it holds that rank*$(B(q)) = n$. *Moreover, without loss of generality, the input matrix is* $B(q) = I_n$.

Step 1: Fully-actuated virtual control system

By proposition 3.6, it is known that (3.36) is a virtual system for (3.23). Thus, under Assumption 4.1, it is a virtual system to the time-varying system given by

$$\begin{aligned} \dot{x}_v &= \begin{bmatrix} 0_n & I_n \\ -I_n & -(E(x)+D(q)) \end{bmatrix} \begin{bmatrix} \frac{\partial H_v}{\partial q_v}(x_v, q) \\ \frac{\partial H_v}{\partial p_v}(x_v, q) \end{bmatrix} + \begin{bmatrix} 0_n \\ I_n \end{bmatrix} u_v, \\ y_v &= \begin{bmatrix} 0_n & I_n \end{bmatrix} \begin{bmatrix} \frac{\partial H_v}{\partial q_v}(x_v, q) \\ \frac{\partial H_v}{\partial p_v}(x_v, q) \end{bmatrix}. \end{aligned} \tag{4.1}$$

with state $x_v = (q_v, p_v)$, parametrized by every trajectory $x(t) = (q(t), p(t))$ of (3.23).

Step 2: Contraction-based control design of the virtual mechanical system

In the following proposition a constructive design procedure is presented where the differentially passive system (2.24) in Lemma 2.13 is taken as target dynamics. This results in a contraction-based backstepping method with differential passivity interpretation.

Proposition 4.2. *Consider system* (3.23) *and the virtual system* (4.1). *Let* $x_d = (q_d, p_d)$ *be a smooth desired trajectory for* (4.1) *on* $\mathcal{X} = T\mathcal{Q}$. *Let us introduce the following error coordinates*

$$\tilde{x}_v := \begin{bmatrix} \tilde{q}_v \\ \sigma_v \end{bmatrix} = \begin{bmatrix} q_v - q_d \\ p_v - p_{\ell r}(\tilde{q}_v, t) \end{bmatrix}, \tag{4.2}$$

where the auxiliary momentum reference function $p_{vr}(\tilde{q}_v, t)$ *is given by*

$$p_r(\tilde{q}_v, t) := M(q)(\dot{q}_d - \phi(\tilde{q}_v) + \overline{v}_r), \tag{4.3}$$

with[1] $\overline{v}_r = 0_n$, $\phi : \mathcal{Q} \to T_q\mathcal{Q}$ *is such that* $\phi(0_n) = 0_n$. *Let* $\Pi_{\tilde{q}_v} : \mathcal{Q} \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n\times n}$ *be a time-varying positive definite Riemannian metric tensor satisfying the inequality*

$$\dot{\Pi}_{\tilde{q}_v,}(\tilde{q}_v, t) - \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v) - \frac{\partial \phi^\top}{\partial \tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}(\tilde{q}_v, t) \leq -2\beta_{q_v}(\tilde{q}_v, t)\Pi_{\tilde{q}_v}(\tilde{q}_v, t), \tag{4.4}$$

with $\beta(\tilde{q}_v, t) > 0$, *uniformly. Assume also that the i-th row of the metric tensor* $\Pi_{\tilde{q}_v,}(\tilde{q}_v, t)$ *is a conservative vector field*[2]. *Consider the x-parametrized composite control given by*

$$u_v(x_v, x, t) := u_v^{ff}(x_v, x, t) + u_v^{fb}(x_v, x, t) + \omega, \tag{4.5}$$

[1] The term $\overline{v}_r$ this chapter is not used. However, it is added explicitly for its usage in (4.2) in Chapter 5.
[2] This ensures that the integral in (4.6) is well defined and independent of the path connecting 0 and $\tilde{q}_v$.

with

$$
\begin{aligned}
u_v^{ff}(x_v, x, t) &= \dot{p}_{\ell r}(\tilde{q}_v, t) + \frac{\partial P_v}{\partial q_v}(q_v) + [E(x) + D(q)]M^{-1}(q)p_{\ell r}(\tilde{q}_v, t), \\
u_v^{fb}(x_v, x, t) &= -\int_0^{\tilde{q}_v} \Pi_{\tilde{q}_v}(\xi, t)d\xi - K_d M^{-1}(q)\sigma_v,
\end{aligned}
\tag{4.6}
$$

where $K_d > 0$ *and* ω *is an external input. Then, the system* (4.1) *in closed-loop with* (4.5) *is strictly differentially passive from* $\delta\omega$ *to* $\delta\overline{y}_{\sigma_v} = M^{-1}(q)\delta\sigma_v$*, with differential storage function*

$$
W(\tilde{x}_v, \delta\tilde{x}_v, q, t) = \frac{1}{2}\delta\tilde{x}_v^\top \begin{bmatrix} \Pi_{\tilde{q}_v}(\tilde{q}_v, t) & 0_n \\ 0_n & M^{-1}(q) \end{bmatrix} \delta\tilde{x}_v. \tag{4.7}
$$

Proof: First step, consider the position dynamics in (4.1) with $y_{\tilde{q}_v} = \tilde{q}_v$ as output and p_v as "input". Define the control-like input as $p_v = \sigma_v + p_{vr}(\tilde{q}, t)$, where σ_v is a new input and $p_{vr}(\tilde{q}_v, t)$ as in (4.3). This results in the "closed-loop" position dynamics

$$
\begin{cases}
\dot{q}_v = M^{-1}(q_d)p_d - \phi(q_v - q_d) + M^{-1}(q)\sigma_v, \\
y_{\tilde{q}_v} = q_v - q_d(t),
\end{cases}
\tag{4.8}
$$

whose prolonged system, in coordinates (4.2), is given by

$$
\tilde{\Sigma}^\delta_{\sigma_v} : \begin{cases}
\dot{\tilde{q}}_v = -\phi(\tilde{q}_v) + M^{-1}(q)\sigma_v \\
\delta\dot{\tilde{q}}_v = -\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v)\delta\tilde{q}_v + M^{-1}(q)\delta\sigma_v \\
y_{\tilde{q}_v} = \tilde{q}_v, \\
\delta y_{\tilde{q}_v} = \delta\tilde{q}_v.
\end{cases}
\tag{4.9}
$$

Take the candidate differential Lyapunov function for system (4.9) to

$$
W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) = \frac{1}{2}\delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\delta\tilde{q}_v. \tag{4.10}
$$

The time derivative of (4.10) along solutions of $\tilde{\Sigma}^\delta_\sigma$ is

$$
\begin{aligned}
\dot{W}_{\tilde{q}_v} &= -\delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v)\delta\tilde{q}_v + \underbrace{\delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)M^{-1}(q)}_{\delta\overline{y}^\top_{\tilde{q}_v}}\delta\sigma + \frac{1}{2}\delta\tilde{q}_v^\top \dot{\Pi}_{q_v}(\tilde{q}_v, t)\delta\tilde{q}_v, \\
&= \frac{1}{2}\delta\tilde{q}_v^\top \left[\dot{\Pi}_{\tilde{q}_v}(\tilde{q}_v, t) - \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v) - \frac{\partial\phi^\top}{\partial\tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}(\tilde{q}_v, t)\right]\delta\tilde{q}_v + \delta\overline{y}^\top_{\tilde{q}_v}\delta\sigma_v.
\end{aligned}
\tag{4.11}
$$

where $\delta\overline{y}_{\tilde{q}_v}^\top = \delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t) M^{-1}(q)$. By (4.4), it follows that

$$\dot{W}_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) \leq -2\beta_{q_v}(\tilde{q}_v, t) W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) + \delta\overline{y}_{\tilde{q}_v}^\top \delta\sigma_v. \tag{4.12}$$

Hence, system (4.8) is (strictly) differentially passive with differential input-output pair $(\delta\sigma_v, \delta\overline{y}_{\tilde{q}_v})$ and differential storage function (4.10). This implies contraction when $\delta\sigma_v = 0_n$; furthermore, convergence to $\tilde{q}_v = 0_n$ if the "input" $\sigma = 0_n$.
For the second step, consider (4.1) with output $y_{\sigma_v} = p_v - p_{\ell r}(\tilde{q}_v, t)$. System (4.1) in coordinates (4.2) is expressed as the system composed by the $\tilde{q}_v$-dynamics in (4.9) and

$$\dot{\sigma}_v = -\frac{\partial P}{\partial q_v}(q_v) - [E(x) + D(x)]\, M^{-1}(q)(\sigma_v + p_{\ell r}) + u_v - \dot{p}_{\ell r}(\tilde{q}_v, t), \tag{4.13}$$

Substitution of the control action $u_v^{ff}(x_v, x, t)$ in (4.6) results in

$$\dot{\sigma}_v = -\left[E(x) + D(q)\right] M^{-1}(q)\sigma_v + u_v^{fb}. \tag{4.14}$$

Notice that above substitution lets us to impose $\sigma_v = 0_n$ as a particular solution of (4.14) when $u_v^{fb} = 0_n$, as desired. Thus, the prolongation of (4.1) in coordinates (4.2) is the system formed by $\tilde{\Sigma}_{\sigma_v}^\delta$ in (4.9) and

$$\tilde{\Sigma}_{u_v^{fb}}^\delta : \begin{cases} \dot{\sigma}_v = -\left[E(x) + D(q)\right] M^{-1}(q)\sigma_v + u_v^{fb}, \\ \delta\dot{\sigma}_v = -\left[E(x) + D(q)\right] M^{-1}(q)\delta\sigma_v + \delta u_v^{fb}, \\ y_{\sigma_v} = \sigma_v, \\ \delta y_{\sigma_v} = \delta\sigma_v, \end{cases} \tag{4.15}$$

Now, let (4.7) be the candidate differential Lyapunov function for the prolonged system $\tilde{\Sigma}_{v\sigma}^\delta$-$\tilde{\Sigma}_{u_v^{fb}}^\delta$ and consider the definition of the control action u_v^{fb} in (4.6). Then

$$\begin{aligned} \dot{W} &= \dot{W}_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) + \frac{1}{2}\delta\sigma_v^\top \dot{M}^{-1}(q)\delta\sigma_v \\ &\qquad + \delta\sigma_v^\top M^{-1}(q)\Big[-\left[E(x) + D(q)\right] M^{-1}(q)\delta\sigma_v + \delta u_v^{fb}, \Big], \\ &\leq -2\beta_{q_v} W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) + \delta\overline{y}_{\tilde{q}_v}^\top \delta\sigma_v + \delta\sigma_v^\top M^{-1}(q)\Big[-D(q)M^{-1}(q)\delta\sigma_v + \delta u_v^{fb}\Big], \\ &= -2\beta_{q_v} W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) + \delta\overline{y}_{\tilde{q}_v}^\top \delta\sigma_v + \delta\sigma_v^\top M^{-1}(q)\Big[-D(q)M^{-1}(q)\delta\sigma_v \\ &\qquad - \Pi_{\tilde{q}_v}\delta\tilde{q}_v - K_d M^{-1}\delta\sigma_v + \delta\omega\Big], \\ &\leq -2\min\{\beta_{q_v}(\tilde{q}_v, t), \lambda_{\min}\{D(q) + K_d\}\lambda_{\min}\{M^{-1}(q)\}\} W(\tilde{q}_v, \delta\tilde{q}_v, t) + \delta\overline{y}_{\sigma_v}^\top \delta\omega. \end{aligned} \tag{4.16}$$

It follows that the conditions in Lemma 2.13 are satisfied and the transformation $\delta\tilde{x}_v = \Theta(x_v, t)\delta x_v$ exists. Therefore, the closed-loop prolonged virtual system is differentially passive. ■

For sake of completeness, the virtual system (4.1) in closed-loop with the control scheme (4.5)-(4.6) is presented. That is, the system given by

$$\begin{bmatrix} \dot{\tilde{q}}_v \\ \dot{\sigma}_v \end{bmatrix} = \begin{bmatrix} -\phi(\tilde{q}_v) + M^{-1}(q)\sigma_v \\ -\int_0^{\tilde{q}_v} \Pi_{q_v}(\tilde{q}_v, t)\mathrm{d}\xi_v - A(x)M^{-1}(q)\sigma_v \end{bmatrix} + \begin{bmatrix} 0_n \\ I_n \end{bmatrix}\omega, \tag{4.17}$$

with matrix $A(x) := E(x) + D(q) + K_d$. The variational system of (4.17) is given by

$$\begin{bmatrix} \delta\dot{\tilde{q}}_v \\ \delta\dot{\sigma}_v \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v)\Pi^{-1}_{\tilde{q}_v}(\tilde{q}_v, t) & I_n \\ -I_n & -A(x) \end{bmatrix}}_{\Xi(\tilde{x},t)-\Upsilon(\tilde{x},t)} \underbrace{\frac{\partial^2 W}{\partial\tilde{x}_v^2}(\tilde{x}_v, \delta\tilde{x}_v, q, t)}_{\Pi(\tilde{x},t)} \begin{bmatrix} \delta\tilde{q}_v \\ \delta\sigma_v \end{bmatrix} + \underbrace{\begin{bmatrix} 0_n \\ I_n \end{bmatrix}}_{\Psi(\tilde{x},t)} \delta\omega. \tag{4.18}$$

Remark 4.3. *It is easy to verify that the prolonged system of the closed-loop system* (4.17) *satisfies Lemma 2.13 with state* $\tilde{x}_v = (\tilde{q}_v, \sigma_v)$.

Step 3: Original mechanical pH system's tracking controller

In the following corollary, it is shown how the result of Proposition 4.2 can be used in order to solve the tracking control problem for the fully-actuated mechanical system (3.23) with $B(q) = I_n$; as described in Subsection 2.2.4.

Corollary 4.4. *Consider the control scheme in* (4.5). *Then, the solutions of system* (3.23) *in closed-loop with the control law*

$$u(x, x, t) = u_v^{ff}(x, x, t) + u_v^{fb}(x, x, t), \tag{4.19}$$

converge exponentially to the trajectory $x_d(t)$ *with rate*

$$\beta(x_v, x, t) = \min\{\beta_{q_v}(q_v, t), \lambda_{\min}\{D(q) + K_d\}\lambda_{\min}\{M^{-1}(q)\}\}, \tag{4.20}$$

where $\lambda_{min}(\cdot)$ *is the minimum eigenvalue of its argument.*

Proof: For $\omega = 0_n$, the origin $x_v = 0_{2n}$ is a solution of the contractive closed-loop virtual system (4.17). Moreover, the original pH system (3.24) in closed-loop with $u(x, x, t)$ in (4.19) defines another solution of (4.17) whenever $x_v = x$, in coordinates (4.2).

Take the external input $\omega = 0_n$. Then, $\delta\omega = 0_n$ in (4.16), and (4.7) qualifies as differential Lyapunov function satisfying

$$W(\tilde{x}_v, \delta\tilde{x}_v, q, t) \leq e^{-2\beta t} W(\tilde{x}_v(0), \delta\tilde{x}_v(0), q, t),$$

for all q and t. This implies that the Finsler's distance in (2.13) between the solutions $x_v = 0_{2n}$ and $x_v = x$ of (4.17), with structure $\mathfrak{F}(\tilde{x}_v, \delta\tilde{x}_v, q, t) = \sqrt{W(\tilde{x}_v, \delta\tilde{x}_v, q, t)}$ shrinks. Indeed, from the previous observations it follows that

$$d(\tilde{x}, 0_n, q, t) = \inf_{\Gamma(\tilde{x}, 0_n)} \int_\gamma \sqrt{W\left(\gamma(s), \frac{\partial\gamma}{\partial s}(s), q, t\right)} ds \leq e^{-\beta t} W(\tilde{x}_v(0), \delta\tilde{x}_v(0), q, t). \quad (4.21)$$

Hence, the state x of the original system (3.24) converges exponentially to the desired solution $x_d(t)$ as $t \to \infty$ with rate $\beta(x_v, x, t)$. ■

4.2 Properties of the closed-loop virtual system

Notice that from a practical point of view, it may be useful to find a different convergence rate $\beta(x_v, x, t)$ in (4.16), which could give more intuition during the gain tunning process. For instance, by Definition 2.4, the differential function (4.7) is bounded as

$$c_1 \|\delta\tilde{x}_v\|^2 \leq W(\tilde{x}_v, \delta\tilde{x}_v, q, t) \leq c_2 \|\delta\tilde{x}_v\|^2, \quad (4.22)$$

with the constants c_1, respectively, c_2 given by

$$c_1 = \min\{\lambda_{\min}(\Pi_{\tilde{q}_v}), \lambda_{\min}(M^{-1})\}, \qquad c_2 = \max\{\lambda_{\max}(\Pi_{\tilde{q}_v}), \lambda_{\max}(M^{-1})\}, \quad (4.23)$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the argument's minimum and maximum eigenvalue, respectively. Then, the derivative of (4.7) in (4.16) can alternatively be bounded as

$$\dot{W}(\tilde{x}_v, \delta\tilde{x}_v, t) \leq -\beta_1(x_v, x, t) W(\tilde{x}_v, \delta\tilde{x}_v, t) + \delta\overline{y}_{\sigma_v}^\top \delta\omega. \quad (4.24)$$

with

$$\beta_1(x_v, x, t) = \frac{c_3}{c_2}, \qquad c_3 = \min\left\{\frac{1}{2}\beta_{q_v} \lambda_{\min}(\Pi_{\tilde{q}_v}), \lambda_{\min}(M^{-1}(D + K_d)M^{-1})\right\}. \quad (4.25)$$

A number of consequences and facts of Proposition 4.2 are presented below.

Corollary 4.5. *Assume there exists a potential function* $P_v : \mathcal{Q} \to \mathbb{R}$ *such that* $\phi(\tilde{q}) =$

$\frac{\partial P_v}{\partial \tilde{q}_v}(\tilde{q})$. *If the metric tensor* $\Pi_{\tilde{q}_v}(\tilde{q}_v, t) = \frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)$*, then the virtual system* (3.36) *in closed-loop with* (4.5) *preserves the pH-like form in* (3.40)*, with the following specifications*

$$J_v(x) = \begin{bmatrix} 0_n & I_n \\ -I_n & -S_H(x) \end{bmatrix}, \quad R_v(x) = \begin{bmatrix} I_n & 0_n \\ 0_n & (D(q) + K_d - \frac{1}{2}\dot{M}(q) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}, \tag{4.26}$$

and (error) Hamiltonian function

$$\tilde{H}(\tilde{x}_v, x) = \frac{1}{2}\sigma_v^\top M^{-1}(q)\sigma_v + \tilde{P}_v(\tilde{q}_v) \tag{4.27}$$

for every trajectory $x(t)$*. Moreover, the map* $\omega \mapsto \tilde{y}_{\sigma_v} = \frac{\partial \tilde{H}}{\partial \sigma_v}(\tilde{x}_v, x)$ *is cyclo-passive*[3] *with respect to the storage function given by* (4.27).

Proof: Consider the closed-loop virtual system in (4.17). By taking the metric tensor as $\Pi_{\tilde{q}_v}(\tilde{q}_v) = \frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)$ and the specifications (4.26), the system (4.17) is expressed as

$$\begin{bmatrix} \dot{\tilde{q}}_v \\ \dot{\sigma}_v \end{bmatrix} = [J_v(x) - R_v(x)] \frac{\partial \tilde{H}}{\partial \tilde{x}_v}(\tilde{x}_v, \tilde{x}) + \begin{bmatrix} 0_n \\ I_n \end{bmatrix}\omega. \tag{4.28}$$

Straightforward computations show that prolonged system associated to (4.28) preserves the structure of system (3.40). Let us now take (4.27) as storage function which satisfies

$$\begin{aligned} \dot{\tilde{H}}(\tilde{x}_v, x) &= \frac{\partial \tilde{H}^\top}{\partial \tilde{x}_v}(\tilde{x}_v, x)\,[J_v(x) - R_v(x)]\,\frac{\partial \tilde{H}}{\partial \tilde{x}_v}(\tilde{x}_v, x) + \frac{\partial \tilde{H}^\top}{\partial \tilde{x}_v}(\tilde{x}_v, x)\begin{bmatrix} 0_n \\ I_n \end{bmatrix}\omega, \\ &= -\frac{\partial \tilde{H}^\top}{\partial \tilde{x}_v}(\tilde{x}_v, x)\begin{bmatrix} I_n & 0_n \\ 0_n & (D(q) + K_d) \end{bmatrix}\frac{\partial \tilde{H}}{\partial \tilde{x}_v}(\tilde{x}_v, x) + \tilde{y}_{\sigma_v}^\top\omega \leq \tilde{y}_{\sigma_v}^\top\omega, \end{aligned} \tag{4.29}$$

which completes the proof of cyclo-passivity of the map $\omega \mapsto \tilde{y}_{\sigma_v}$. ■

This result tells us that, under the hypotheses of Corollary 4.5, if we take as target closed-loop system to (4.17), then it may not be necessary to invoke differential analysis arguments, since standard stability and passivity are still applicable. Moreover, this is closely related to the structure preserving passivity-based methods of mechanical pH system as IDA-PBC (Ortega et al. 2002). For instance, if $\phi(\tilde{q}_v) = -\Lambda\tilde{q}_v$ we recover the scheme in (Reyes-Báez et al. 2017) which has a similar structure as the schemes in (Dirksz and Scherpen 2010) and (Romero, Ortega and Sarras 2015), modulus a GCT. We point out that in our approach such GCT is not necessary.

[3] System (2.1) is said cyclo-passive if there esists $S : \mathcal{X} \to \mathbb{R}$ satisfying $\dot{S}(x) \leq y^\top u$. This class of systems are distinguished from the standard passive systems, where $S(x)$ is positive semi-definite (Willems 1972).

Remark 4.6. *The* timed-*IDA-PBC method proposed in (Yaghmaei and Yazdanpanah 2017) uses the contraction property of a target pH system for solving the tracking problem of a desired reference. In particular, one of the main assumptions is that the interconnection and dissipation matrices are constant. In this regards, our proposed method v-CBC for mechanical pH systems relaxes this assumption, since by employing virtual systems, such matrices are allowed to be time-varying as in system* (4.17)*. Thus, the timed-IDA-PBC design can be employed in the step 2 in the v-CBC methodology described in Section 2.2.3.*

In the following result conditions on the metric tensor $\Pi_{\tilde{q}_v}(\tilde{q}_v, t)$ are relaxed with respect to the previous corollary. Sufficient conditions for preserving the structure of the variational dynamics (3.42), instead of in the original system itself as in Corollary 4.5.

Corollary 4.7. *Consider the virtual system* (3.36) *in closed-loop with* (4.5)*. Let the metric tensor $\Pi_{\tilde{q}_v}(\tilde{q}_v, t)$ and the vector field $\phi(\tilde{q}_v)$ be such that*

$$\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}^{-1}(\tilde{q}_v, t) = \left(\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}^{-1}(\tilde{q}_v, t)\right)^\top . \tag{4.30}$$

Then, the closed-loop variational dynamics preserves the structure as in (3.42) *with matrices*

$$\begin{aligned} \frac{\partial^2 \tilde{H}_v}{\partial \tilde{x}_v^2}(\tilde{x}_v, x) &= \frac{\partial^2 W}{\partial \tilde{x}_v^2}(\tilde{x}_v, \delta\tilde{x}_v, q, t), \\ R_v(\tilde{x}_v, x, t) &= \operatorname{diag}\left\{\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}^{-1}(\tilde{q}_v, t), D(q) + K_d - \frac{1}{2}\dot{M}(q)\right\}, \\ J_v(\tilde{x}_v, x, t) &= \begin{bmatrix} 0_n & I_n \\ -I_n & -S_H(x) \end{bmatrix}, \quad g(x) = \begin{bmatrix} 0_n \\ I_n \end{bmatrix}. \end{aligned} \tag{4.31}$$

Proof: Substitute hypothesis (4.30) and definitions in (4.31) in the closed-loop variational system (4.18). Straightforward computations show that matrices in (4.31) satisfy the properties of those in (3.41). This completes the proof. ■

The inequality in (4.4) of Proposition 4.2 implies *incremental passivity* of the static mapping $\phi(\tilde{q}_v)$. In the following proposition sufficient conditions on $\phi(\tilde{q}_v)$ are given in order to guarantee strict (differential) passivity of the closed-loop system (4.17) simultaneously. Recall the definition of incremental passivity (Pavlov and Marconi 2006).

Definition 4.8. *The map $\chi(z)$ is incrementally passive if it satisfies the monotonicity condition:*

$$[\chi(z_2) - \chi(z_1)]^\top (z_2 - z_1) \geq 0, \tag{4.32}$$

for any z_1 and z_2. The property is strict if the inequality (4.32) *is strict for $z_1 \neq z_2$.*

Corollary 4.9. *Consider system* (4.1) *in closed-loop with* (4.5). *If $\Pi_{\tilde{q}_v}(\tilde{q}_v, t)$ in* (4.4) *is constant. Then, the map $\chi(\tilde{q}_v) = \Pi_{\tilde{q}_v}\phi(\tilde{q}_v)$ is strictly incrementally passive.*

Proof: Suppose $\Pi_{\tilde{q}_v}$ in (4.4) is constant. Then

$$\Pi_{\tilde{q}_v}\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v) + \frac{\partial\phi^\top}{\partial\tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v} \geq 2\beta_{q_v}\Pi_{\tilde{q}_v}. \tag{4.33}$$

Let $\gamma_f(s) = \tilde{q}_{v1} + s(\tilde{q}_{v2} - \tilde{q}_{v1})$ be a path joining the points $\tilde{q}_{v1}, \tilde{q}_{v2} \in \mathcal{Q}$, $s \in [0, 1]$. Notice that

$$\chi(\tilde{q}_{v2}) - \chi(\tilde{q}_{v1}) = \int_0^1 \Pi_{\tilde{q}_v}\frac{\partial\phi}{\partial s}(\gamma_f(s))ds = \int_0^1 \Pi_{\tilde{q}_v}\frac{\partial\phi}{\partial\tilde{q}_v}(\gamma_f(s))(\tilde{q}_{v2} - \tilde{q}_{v1})ds, \tag{4.34}$$

after substitution of this in (4.32), it follows that

$$\begin{aligned}(\tilde{q}_{v2} - \tilde{q}_{v1})^\top\left[\chi(\tilde{q}_{v2}) - \chi(\tilde{q}_{v1})\right] &= \int_0^1 (\tilde{q}_{v2} - \tilde{q}_{v1})^\top\Pi_{\tilde{q}_v}\frac{\partial\phi}{\partial\tilde{q}_v}(\gamma_f(s))(\tilde{q}_{v2} - \tilde{q}_{v1})ds,\\ &= \int_0^1 (\tilde{q}_{v2} - \tilde{q}_{v1})^\top\left[\Pi_{\tilde{q}_v}\frac{\partial\phi}{\partial\tilde{q}_v} + \frac{\partial\phi^\top}{\partial\tilde{q}_v}\Pi_{\tilde{q}_v}\right](\tilde{q}_{v2} - \tilde{q}_{v1})ds.\end{aligned} \tag{4.35}$$

Then, we have that

$$(\tilde{q}_{v2} - \tilde{q}_{v1})^\top\left[\chi(\tilde{q}_{v2}) - \chi(\tilde{q}_{v1})\right] \geq 2\beta_{\tilde{q}_v}(\tilde{q}_{v2} - \tilde{q}_{v1})^\top\Pi_{\tilde{q}_v}(\tilde{q}_{v2} - \tilde{q}_{v1}) > 0, \tag{4.36}$$

which proves that $\chi(\tilde{q}_v) = \Pi_{\tilde{q}_v}\phi(\tilde{q}_v)$ is incrementally passive. Finally, taking $\Pi_{\tilde{q}_v} = I_n$ completes the proof. ■

As said before, conditions in Corollary 4.9 are only sufficient for the incremental stability property of the map $\phi(\tilde{q}_v)$. However, there may exist incrementally passive maps which do not satisfy the inequality (4.4). The following result gives necessary and sufficient conditions on $\phi(\tilde{q}_v)$ in order to guarantee both properties, simultaneously.

Proposition 4.10. *Consider the maps $\chi(\tilde{q}_v) = \Pi_{\tilde{q}_v}\phi(\tilde{q}_v)$ with $\Pi_{\tilde{q}_v}$ a positive definite and constant metric tensor. Then, inequality* (4.4) *is satisfied if and only if the following condition holds:*

$$(\tilde{q}_{v2} - \tilde{q}_{v1})^\top\left[\chi(\tilde{q}_{v2}) - \chi(\tilde{q}_{v1})\right] \geq 2\beta_{\tilde{q}_v}(\tilde{q}_{v2} - \tilde{q}_{v1})^\top\Pi_{\tilde{q}_v}(\tilde{q}_{v2} - \tilde{q}_{v1}) > 0. \tag{4.37}$$

Proof: The sufficiency follows from Lemma 4.9. For the converse, assume that (4.37) holds. Then, from identity (4.35) and (4.37) the following is true

$$\begin{aligned}\int_0^1 (\tilde{q}_{v2}-\tilde{q}_{v1})^\top \left(\Pi_q \frac{\partial\phi}{\partial\tilde{q}_v} + \frac{\partial\phi^\top}{\partial\tilde{q}_v}\Pi_k\right)(\tilde{q}_{v2}-\tilde{q}_{v1})ds \\ \geq \int_0^1 2\beta_{\tilde{q}_v}(\tilde{q}_{v2}-\tilde{q}_{v1})^\top \Pi_k (\tilde{q}_{v2}-\tilde{q}_{v1})ds > 0.\end{aligned} \tag{4.38}$$

The monotonicity property of integrals in (4.38) implies that inequality (4.4) holds. ■

Remark 4.11 (Contraction of the virtual system is necessary)**.** *If the conditions of Proposition* (4.10) *are not satisfied, then using Corollary 4.9 we still can find the (incrementally) passive map χ that make* (4.7) *a Lyapunov function for system* (4.17) *with minimum at the origin. However, Corollary 4.9 does not ensure that there exist a unique steady-state trajectory of* (4.17) *since contraction condition* (4.4) *is not necessarily satisfied. This in turn implies that, although the system* (3.23) *in closed-loop with the* (4.5) *is stable, it may not converge to the desired q_d.*

A further consequence of Proposition 4.2 related to the idea of controlled-invariant sliding manifolds is presented[4]. To this end, the notions of invariant set and sliding manifold are recalled; for further details see (Sira-Ramírez 2015, Utkin 2013).

Definition 4.12. *The set $\mathcal{S} \subset \mathcal{X}$ is* invariant *for the control system* (2.1) *if whenever $x(t_0) \in \mathcal{S}$ implies that $x(t) \in \mathcal{S}$, for all $t > t_0$.*

Definition 4.13. *A* sliding manifold *for the control system* (2.1) *is a subset $\Omega(t)$ of the state space $\mathcal{X}$ defined as the intersection of n smooth $(N-1)$-dimensional subsets given by*

$$\Omega(t) = \{x \in \mathcal{X} : \sigma(x,t) = [\sigma_1(x,t),\ldots,\sigma_n(x,t)]^\top = 0_n\} \tag{4.39}$$

where $\sigma(x,t)$ is the sliding variable*, with $\sigma_i \in C^\infty(\mathcal{X}\times\mathbb{R})$ for $i \in \{1,\ldots,n\}$.*

It follows that $\Omega(t)$ is locally a $(N-n)$-dimensional sub-manifold of $\mathcal{X}$.

The smooth control vector u_{eq}, known as the *equivalent control*, renders the manifold $\Omega(t)$ invariant for systems (2.1) in the sense of Definition 4.12, see (Utkin 2013). If $\mathrm{rank}\{L_{g_i}\sigma_j\} = n$ for every $x \in \Omega(t)$ and every $i, j \in \{1,\ldots,n\}$, then the equivalent control is the well defined solution to the following invariance conditions

$$\sigma(x,t) = 0, \quad \text{and} \quad \dot{\sigma}(x,t) = 0, \quad \text{for every} \quad t. \tag{4.40}$$

[4]Similar results on invariant manifold stabilization via contraction methods were obtained in (Andrieu et al. 2016), (Wang et al. 2016), and (Manchester and Slotine 2017).

System $\dot{x} = f(x, t) + \sum_{i=1}^{n} g_i(x, t)u_{eq,i}(x, t)$ is describes the *ideal sliding motion* on $\Omega(t)$.

Using sliding manifolds in control design has as goal to desig a suitable control scheme $u = u_{eq} + u_{att}$, such that u_{eq} renders $\Omega(t)$ to be an invariant sliding manifold under invariance conditions (4.40), and u_{att} makes the invariant manifold attractive. Identifying the equivalent control u_{eq} with $u_v^{ff}(x_v, x, t)$ in (4.5) and the attractivity action u_{att} with $u_v^{fb}(x_v, x, t)$ in (4.5), we have following result.

Corollary 4.14. *The sliding manifold*

$$\Omega(t) = \{x_v \in \mathcal{X} : \sigma_v = M(q)[\dot{\tilde{q}}_v + \phi(\tilde{q}_v)] = 0_n\}, \tag{4.41}$$

is invariant and attractive, with ideal sliding motion

$$\dot{q}_v = \frac{\partial H_v}{\partial p_v}(q_v, p_{vr}, t). \tag{4.42}$$

Proof: The control u_{ffv} in (4.6) qualifies as the equivalent control (Sira-Ramírez 2015) that satisfies the invariance conditions $\sigma_v = \dot{\sigma}_v = 0_n$, uniformly in t. This guarantees that when constrained to the submanifold $\sigma_v = 0_n$, all the trajectories will remain in it for all $t > t_0$. By Proposition 4.2, $\sigma_v \to 0_n$, which means that the invariant manifold is also attractive. Finally, since system (3.36) has *regular form*, definitions of p_{vr} and σ_v imply, that the *reduced-order* ideal sliding motion is (4.42) (Sira-Ramírez 2015). ∎

4.3 Experimental closed-loop evaluation

In this section three different controllers that belong to the family described in Proposition 4.1 are constructed. These schemes solve the trajectory tracking problem in a fully-actuated planar robot of two degrees of freedom (DoF). Experimental evaluation of the above schemes is also presented.

4.3.1 Experimental setup

The experimental setup consists of a two DoF planar robot from Quanser[5]. This system can operate either as a rigid or as a flexible-joints robot; see Figure 4.1.

Since the result in Proposition 4.1 is only valid for fully-actuated systems, the rigid

[5](*Quanser Consulting Inc. 2-DOF serial flexible link robot, Reference Manual, Doc. No. 763, Rev. 1, 2008* n.d.).

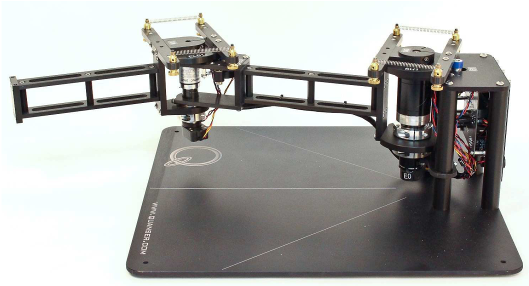

Figure 4.1: Quanser 2 degrees of freedom serial flexible joints robot manipulator.

robot configuration is considered. The corresponding inertia matrix $M_\ell(q_\ell)$ is given by[6]

$$M_\ell(q_\ell) = \begin{bmatrix} a_1 + a_2 + 2b\cos(q_{\ell 2}) & a_2 + b\cos(q_{\ell 2}) \\ a_2 + b\cos(q_{\ell 2}) & a_2 \end{bmatrix}, \tag{4.43}$$

with $a_1 = m_{\ell 1} r_{\ell 1}^2 + m_{\ell 2}\ell_{\ell 1}^2 + I_{\ell 1}, a_2 = m_{\ell 2} r_{\ell 2}^2 + I_{\ell 2}, b = m_{\ell 2}\ell_{\ell 1} r_{\ell 2}$. The robot can also be modeled as (3.26), with

$$E_\ell(q_\ell, p_\ell) = b\sin(q_{\ell 2}) \begin{bmatrix} \dot{q}_{\ell 2} & -\dot{q}_{\ell 1} \\ \dot{q}_{\ell 1} + \dot{q}_{\ell 2} & 0 \end{bmatrix}_{\dot{q}_\ell = M^{-1}(q_\ell)p_\ell}. \tag{4.44}$$

where matrices

$$S_\ell(q_\ell, p_\ell) = b\sin(q_{\ell 2}) \begin{bmatrix} 0_{n_\ell} & -\dot{q}_{\ell 1} - 0.5\dot{q}_{\ell 2} \\ \dot{q}_{\ell 1} + 0.5\dot{q}_{\ell 2} & 0_{n_\ell} \end{bmatrix}, \quad \dot{M}_\ell(q_\ell) = -b\sin(q_{\ell 2}) \begin{bmatrix} 2\dot{q}_{\ell 2} & \dot{q}_{\ell 2} \\ \dot{q}_{\ell 2} & 0_{n_\ell} \end{bmatrix}. \tag{4.45}$$

The physical parameters of this system are shown in Table 4.1:

Parameter	Value	Parameter	Value	Parameter	Value
$m_{\ell 1}$	$1.510kg$	$I_{\ell 1}$	$.039kgm^2$	$\ell_{\ell 1}$	$.343m$
$m_{\ell 2}$	$0.873kg$	$I_{\ell 2}$	$.0081kgm^2$	$\ell_{\ell 2}$	$.267m$
m_{m1}	$0.23kg$	$r_{\ell 1}$	$.159m$	D_ℓ	diag$\{.8, .55\}Ns/m$
m_{m2}	$0.01kg$	$r_{\ell 2}$	$.055m$	D_m	diag$\{.2, 90\}Ns/m$

Table 4.1: Quanser robot physical parameter values.

[6]The subscript ℓ in the state and parameters of the rigid robot is explicitly to emphasize that the robot's dynamics is given by the de *links* equations of motion. This notation will be important in the next chapter.

Desired trajectory

For all the experiments in this chapter, a Bernoulli's lemniscate trajectory in the work space is considered; that is a curve that the robot's end effector must track. The parametric equations of this trajectory on $\mathbb{R}^2$ are given by

$$r = \sqrt{\cos(2t)}, \quad x = 0.4r\sin(t), \quad y = 0.005r\cos(t) + 0.55, \qquad t \in I \subset \mathbb{R}. \tag{4.46}$$

Thus, in order to find the corresponding joints desired trajectory $q_{\ell d}(t)$ for the curve (4.46) in task space of the robot arm is employed. The procedure for computing the inverse kinematics for the RR robot is pretty standard and it is not presented here; the interested reader is referred to (Craig 2009). Since the user interface of the experimental setup runs on a Matlab-Simulink environment, the inverse kinematics algorithm to find $q_{\ell d}(t)$ is implemented internally via symbolic variables.

4.3.2 $(\Lambda_\ell, K_{\ell d}, \Lambda_\ell \tilde{q}_{\ell v})$-controller

The first controller presented here is a structure preserving scheme in the sense of Corollary 4.5. To this end, let $\Pi_{\tilde{q}_{\ell v}}(\tilde{q_{\ell v}}) = \Lambda_\ell$ be a constant metric tensor, where the matrix $\Lambda_\ell = \mathrm{diag}\{\lambda_{\ell 1}, \cdots, \lambda_{\ell n}\}$ is a positive definite matrix and the vector field $\phi_\ell(\tilde{q}_{\ell v}) = \Lambda_\ell \tilde{q}_{\ell v}$. Then, the contraction inequality in (2.17) for this case yields

$$-\Lambda_\ell^2 - \Lambda_\ell^2 \leq -\beta_{q_{\ell v}} \Lambda_\ell \iff -2\Lambda_\ell^2 \leq -\beta_{q_{\ell v}} \Lambda_\ell, \tag{4.47}$$

and by Rayleigh's theorem, the convergence rate with the above specifications in (4.4) is

$$\beta_{\tilde{q}_{\ell v}} = 2\frac{\lambda_{min}(\Lambda_\ell^2)}{\lambda_{max}(\Lambda_\ell)}. \tag{4.48}$$

The closed-loop potential energy function in (4.27) is given by

$$\tilde{P}_{\ell v}(\tilde{q}_v) = \int_0^{\tilde{q}_{\ell v}} \Lambda_\ell \rho_\ell \mathrm{d}\rho_\ell = \frac{1}{2}\tilde{q}_{\ell v}^\top \Lambda_\ell \tilde{q}_{\ell v}.$$

It follows that the controller in Corollary 4.4 in this case takes the form

$$\begin{aligned}
u_{\ell v}(x, x, t) &= u_{\ell v}^{ff}(x, x, t) + u_{\ell v}^{fb}(x, x, t), \\
u_{\ell v}^{ff}(x, x, t) &= \dot{p}_{\ell r}(q, t) + \frac{\partial P_v}{\partial q_v}(q) + [E_\ell(q_\ell, p_\ell) + D_\ell(q)]M_\ell^{-1}(q_\ell)p_{\ell r}, \\
u_{\ell v}^{fb}(x, x, t) &= -\Lambda \tilde{q}_\ell - K_{\ell d} M_\ell^{-1}(q_\ell)\sigma_\ell, \\
p_{\ell r}(q, t) &= M_\ell(q_\ell)(\dot{q}_{\ell d} - \Lambda \tilde{q}_\ell).
\end{aligned} \tag{4.49}$$

This scheme was presented in (Reyes-Báez et al. 2017), and it is closely related to the scheme in (Romero, Ortega and Sarras 2015) based on the IDA-PBC technique. This controller can be interpreted as the Hamiltonian counterpart of the one in (Slotine and Li 1987).

The performance of the closed-loop RR robot with the controller (4.49) is shown in Figure 4.2. For this example the "proportional gain" is $\Lambda_\ell = \operatorname{diag}\{7, 25\}$ and the "derivative gain" is $K_{\ell d} = \operatorname{diag}\{0.2, 0.1\}$. The first gain shapes the "potential energy" of the closed-loop system in error coordinates, whereas the later gain injects damping in the momentum error variable. As it is appreciated in Figure 4.2, after a short transient time the robot's and desired trajectories converge towards each other. Indeed, this is confirmed in the second and third plots, where the position and error time response are shown. Nevertheless, the joint position error convergence is only practical since there is a steady-state error of order 10^{-2}. This is mainly attributed to the noise induced by numerical differentiation to estimate the velocity $\dot{q}_\ell$. The control effort (fourth plot in Figure 4.2) is smooth after the transient.

4.3.3 $(\Lambda_\ell, K_{\ell d}, \Lambda_\ell \mathrm{Tanh}(\tilde{q}_{\ell v}))$-controller

A controller that satisfies hypotheses in Corollary 4.7 is constructed here. This means that the variational closed-loop virtual system in (4.18) preserves the variational pH-like structure (3.42). To this end, the following operators acting on $w \in \mathbb{R}^p$ are defined as

$$\mathrm{Tanh}(w) := \begin{bmatrix} \tanh(w_1), \\ \vdots \\ \tanh(w_p) \end{bmatrix} \in \mathbb{R}^p, \qquad \mathrm{SECH}(w) = \begin{bmatrix} \mathrm{sech}(w_1) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \mathrm{sech}(w_p) \end{bmatrix} \in \mathbb{R}^{p\times p}. \tag{4.50}$$

Consider the proportional (Λ_ℓ) and derivative ($K_{\ell d}$) gain matrices to be chosen as in the previous control scheme. However, take $\phi_\ell(\tilde{q}_{\ell v}) = \Lambda_\ell \mathrm{Tanh}(\tilde{q}_{\ell v})$. Then,

$$\frac{\partial \phi_\ell}{\partial \tilde{q}_{\ell v}}(\tilde{q}_{\ell v}) = \Lambda_\ell \mathrm{SECH}^2(\tilde{q}_{\ell v}), \tag{4.51}$$

where $\mathrm{SECH}(\tilde{q}_{\ell v})$ is bounded as $0_{n\times n} < \mathrm{SECH}(\tilde{q}_{\ell v}) \leq I_n$, due to $\mathrm{sech}(\cdot) \in (0, 1]$. By design, both matrices Λ_ℓ and $\mathrm{SECH}(\cdot)$ are diagonal. This implies that inequality in (4.4) becomes

$$-\Lambda_\ell^2 \mathrm{SECH}^2(\tilde{q}_{\ell v}) - \mathrm{SECH}^2(\tilde{q}_{\ell v})\Lambda_\ell^2 \leq -\beta_{q_{\ell v}}\Lambda_\ell \iff -2\Lambda_\ell^2 \mathrm{SECH}^2(\tilde{q}_{\ell v}) \leq -\beta_{q_{\ell v}}\Lambda_\ell, \tag{4.52}$$

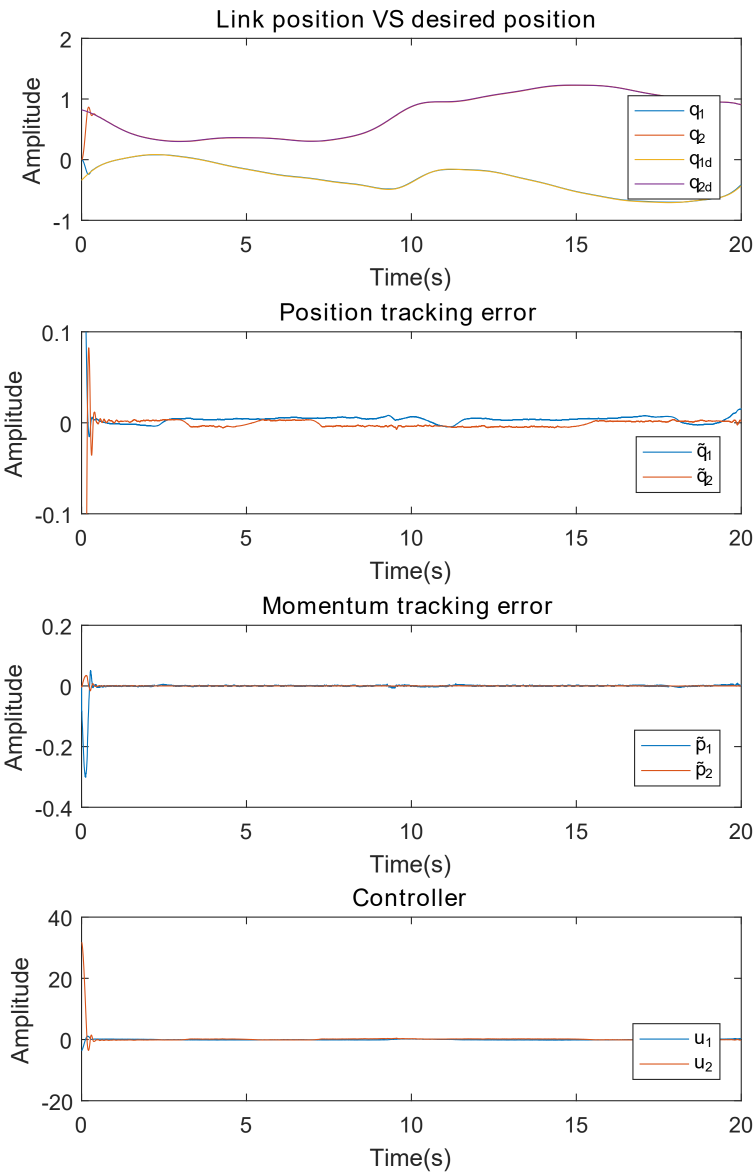

Figure 4.2: Performance of $(\Lambda, K_d, \Lambda\tilde{q}_v)$ trajectory tracking controller.

where the convergence rate is given by

$$\beta_{\tilde{q}_{\ell v}} = 2\frac{\lambda_{min}(\Lambda_\ell^2)\cdot\lambda_{min}(\mathrm{SECH}^2(\tilde{q}_{\ell v}))}{\lambda_{max}(\Lambda_\ell)}. \tag{4.53}$$

Notice that despite the closed-loop virtual system does not preserve the pH-like structure in (3.50), the vector field $\phi_\ell(\tilde{q}_{\ell v}) = \Lambda_\ell \mathrm{Tanh}(\tilde{q}_{\ell v})$ is conservative. Indeed,

$$\tilde{P}_\ell(\tilde{q}_{\ell v}) = \int_0^{\tilde{q}_{\ell v}} \Lambda_\ell \mathrm{Tanh}(\xi)\mathrm{d}\xi = \sum_{k=1}^{n} \lambda_{\ell k} \ln(\cosh(\tilde{q}_{\ell v k})). \tag{4.54}$$

The potential function (4.54), similar to the one in (4.27), can not be longer interpreted as potential energy, since the physical structure is preserved only in the variational system. However, this still is an *artificial coupling force* between q_v and q_d. Thus, following Corollary 4.4 the controller is given as

$$\begin{aligned} u_{\ell v}(x,x,t) &= u_{\ell v}^{ff}(x,x,t) + u_{\ell v}^{fb}(x,x,t),\\ u_{\ell v}^{ff}(x,x,t) &= \dot{p}_{\ell r}(q,t) + \frac{\partial P_v}{\partial q_v}(q) + [E_\ell(q_\ell,p_\ell) + D_\ell(q)]M_\ell^{-1}(q_\ell)p_{\ell r},\\ u_{\ell v}^{fb}(x,x,t) &= -\Lambda\tilde{q}_\ell - K_{\ell d}M_\ell^{-1}(q_\ell)\sigma_\ell,\\ p_{\ell r}(q,t) &= M_\ell(q_\ell)(\dot{q}_{\ell d} - \Lambda_\ell \mathrm{Tanh}(\tilde{q}_{\ell v})). \end{aligned} \tag{4.55}$$

This control scheme is very close to the previous one since the feedback action $u_{\ell v}^{fb}(x,x,t)$ is the same. However, the feedforward-like term $u_{\ell v}^{ff}(x,x,t)$ makes the closed-loop ideal sliding dynamics in (4.42) to be driven by the saturated action $p_{\ell r}(q,t)$ in (4.55).

To apply this control scheme, as in the previous controller, take the "proportional" and "derivative" gains as $\Lambda_\ell = \mathrm{diag}\{7, 25\}$ and $K_{\ell d} = \mathrm{diag}\{0.2, 0.1\}$, respectively. The performance is shown in Figure 4.3, where the convergence behavior is close one as shown in Figure 4.2. This is clear from the relation between (4.52) and (4.47) due to the image of sech$(\cdot)$ is the set $[0, 1)$.

4.3.4 $(\Lambda, K_d, \phi_{\mu_1}(\cdot))$-controller

Another approach for constructing the metric tensor $\Pi_{\tilde{q}_{\ell v}}(\tilde{q_{\ell v}}) = \Lambda_\ell$ and the vector field $\phi(\cdot)$ is exploiting the relation between the Riemannian and the matrix measure contraction frameworks by means of the inequality in (2.21). To this end, from Table 2.1 consider the matrix measure with respect to the $\|\cdot\|_1$ norm defined as

$$\mu_1(\overline{J}(\tilde{q}_v)) := \max_j \left(\overline{J}_{jj}(\tilde{q}_v) + \sum_{i\neq j} |\overline{J}_{ij}(\tilde{q}_v)| \right), \qquad \overline{J}(\tilde{q}_v) = \frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v). \tag{4.56}$$

Then, given the positive definite constant matrix defined as $\Theta = \mathrm{diag}\{\theta_1, \theta_2\}$, the metric tensor satisfies $\Lambda = \Theta^\top\Theta$ and condition (4.4) is equivalent to $\mu_1(\overline{J}(\tilde{q}_v, q, t)) \leq -2\beta_{\tilde{q}_v}$, where the convergence rate is $2\beta_{\tilde{q}_v} = \min\{c_1^2, c_2^2\}$, with positive constants c_1^2 and c_2^2 satis-

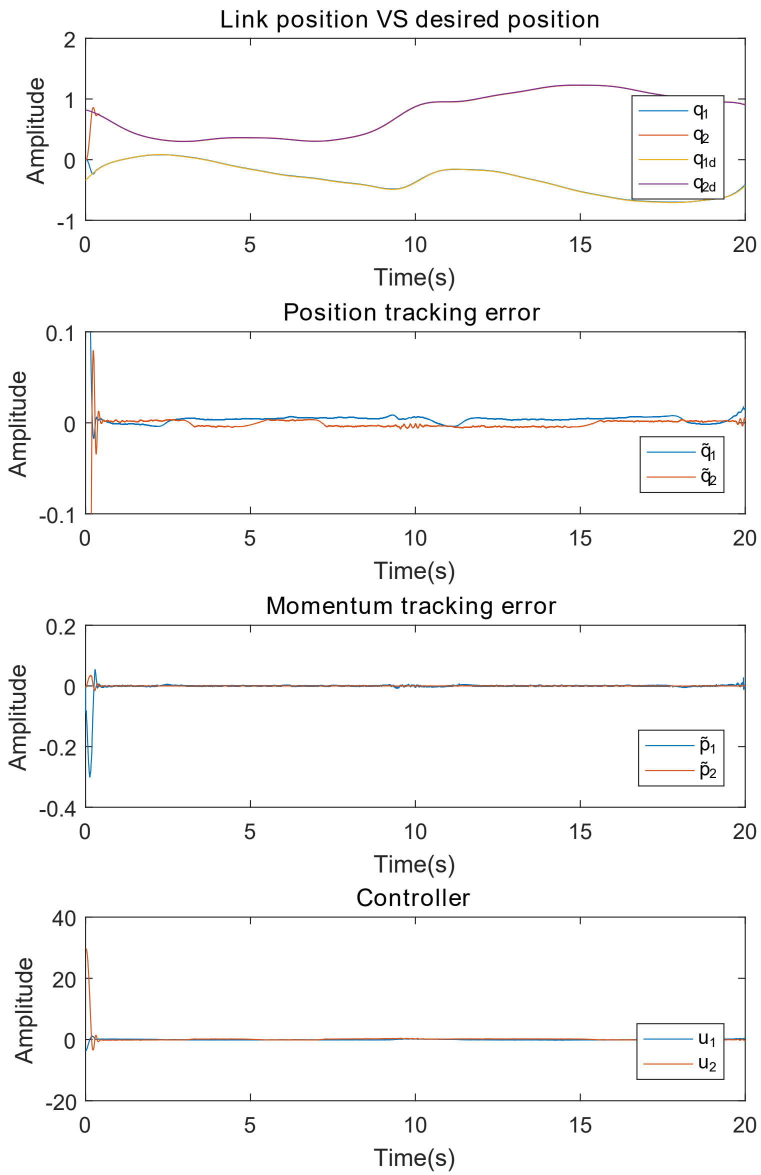

Figure 4.3: Performance of $(\Lambda, K_d, \Lambda\mathrm{Tanh}(\tilde{q}_v))$ trajectory tracking controller.

fying

$$\bar{J}_{11}(\tilde{q}_v, q, t) + |\bar{J}_{21}(\tilde{q}_v, q, t)| < -c_1^2 \quad \text{and} \quad \bar{J}_{22}(\tilde{q}_v, q, t) + |\bar{J}_{12}(\tilde{q}_v, q, t)| < -c_2^2. \tag{4.57}$$

For the above specifications, the matrix measure is explicitly given by

$$\mu_1(\bar{J}) = \max\left\{-\frac{\partial\phi_1}{\partial\tilde{q}_{v1}} + \left|\frac{\theta_2}{\theta_1}\frac{\partial\phi_2}{\partial\tilde{q}_{v1}}\right|, -\frac{\partial\phi_2}{\partial\tilde{q}_{v2}} + \left|\frac{\theta_1}{\theta_2}\frac{\partial\phi_1}{\partial\tilde{q}_{v2}}\right|\right\}. \tag{4.58}$$

where $\phi_1(\tilde{q}_v)$ and $\phi_2(\tilde{q}_v)$ are the components of

$$\phi(\tilde{q}_v) = \begin{bmatrix}(1+\kappa_1)\tilde{q}_{v1} + \frac{\theta_2}{\theta_1}\tanh(\tilde{q}_{v2})\\ \frac{\theta_1}{\theta_2}\tanh(\tilde{q}_{v1}) + (1+\kappa_2)\tilde{q}_{v2}\end{bmatrix} =: \phi_{\mu_1}(\tilde{q}_v), \tag{4.59}$$

and $c_1^2 = \kappa_1 > 0$, $c_2^2 = \kappa_2 > 0$. Indeed, by noticing that the range of $\mathrm{sech}(\cdot)$ is $(0,1]$, it is straightforward to verify conditions (4.57) for (4.59), namely

$$-(1+\kappa_1) + \left|\mathrm{sech}^2(\tilde{q}_{v1})\right| \leq -\kappa_1 = -c_1^2; \qquad -(1+\kappa_2) + \left|\mathrm{sech}^2(\tilde{q}_{v2})\right| \leq -\kappa_2 = -c_2^2. \tag{4.60}$$

It follows that the control scheme is given by

$$\begin{aligned} u_{\ell v}(x,x,t) &= u_{\ell v}^{ff}(x,x,t) + u_{\ell v}^{fb}(x,x,t),\\ u_{\ell v}^{ff}(x,x,t) &= \dot{p}_{\ell r}(q,t) + \frac{\partial P_v}{\partial q_v}(q) + [E_\ell(q_\ell,p_\ell) + D_\ell(q)]M_\ell^{-1}(q_\ell)p_{\ell r},\\ u_{\ell v}^{fb}(x,x,t) &= -\Lambda\tilde{q}_\ell - K_{\ell d}M_\ell^{-1}(q_\ell)\sigma_\ell,\\ p_{\ell r}(q,t) &= M_\ell(q_\ell)(\dot{q}_{\ell d} - \phi_{\mu_1}(\tilde{q}_v)). \end{aligned} \tag{4.61}$$

With this scheme, neither the pH-like (3.40) nor its variational dynamics (3.42) structures are preserved. Nevertheless, uniform global exponential convergence is guarantee by Corollary 4.4. Moreover, with the controller (4.61) the convergence rate $2\beta_{\tilde{q}_v} = \min\{c_1^2, c_2^2\}$ does not depend on the proportional gain Λ.

In order to compare the closed-loop performance with respect to the controllers (4.49) and (4.55), pick the gains $\kappa_1 = 17, \kappa_2 = 9$, $\theta_1 = \sqrt{\lambda_1}$, $\theta_2 = \sqrt{\lambda_2}$ with λ_1, λ_2 and $K_{\ell d}$ as before. Figure 4.4 shows the time response, where the convergence is reached. However, an improvement of this controller performance is not appreciated with respect to the previous two schemes since the same proportional and derivative gains were taken.

4.4 Conclusions and future research

4.4.1 Conclusions

The virtual contraction based control method has been used to solve the tracking problem for fully-actuated mechanical pH systems. Thanks to the energy-based properties of the

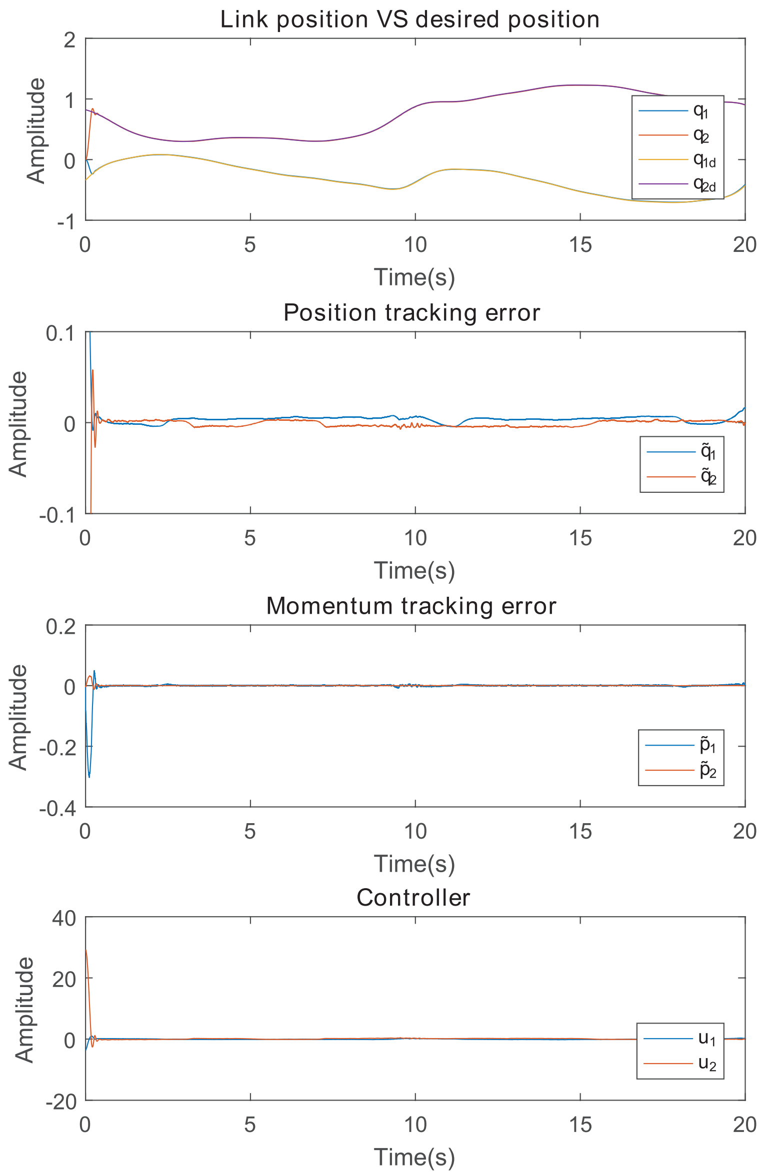

Figure 4.4: Performance of $(\Lambda, K_d, \phi_{\mu_1}(\cdot))$ trajectory tracking controller.

virtual system (4.1) (see Chapter 3) it is possible to have a physical insight of the control scheme (4.5)-(4.6) under the different[7] functions $\phi(\cdot)$. Moreover, taking advantage of the virtual system's rectangular form, the controller was constructed in a recursively manner, similar to a backsttepping but in the variational virtual system.

Three different control algorithms for a rigid robot manipulator of two degrees of freedom have been constructed; each of which has different structural properties. Experimental results confirm the theoretical developments.

4.4.2 Future research

It remains open to explore other constructive control procedures by employing different norms, as those in Table 2.1, or numerical approaches for Riemannian contraction matrix construction. These may offer and improvement in the time response of the closed-loop systems. Furthermore, the matrix measure approach can be used for ensuring contraction of the complete virtual system with state (q_v, p_v), rather than only on q_v dynamics as it was done here. This, however, implies that the closed-loop structure preserving properties are going to be lost.

From a practical point of view, it is desirable that the control scheme (4.5)-(4.6) does not require the velocity measurements to be available as in (Dirksz and Scherpen 2010). Thus, it is an open problem to reformulated the contraction based scheme such that velocity measurements are not required for ensuring closed-loop exponential stability. An output feedback setting has been explored in (Yaghmaei and Yazdanpanah 2018).

Finally, the the Hamiltonian counterpart of the adaptive tracking controller for fully-actuated EL systems in (Slotine and Li 1987) is also an open problem[8].

[7] A number of algorithms that lie in the controllers family in (4.5)-(4.6) were constructed and experimentally evaluated as part of the master thesis project (Pan 2018).

[8] An adaptive controller for mechanical pH systems was introduced in (Dirksz and Scherpen 2010).

Chapter 5

Virtual contraction based control of flexible-joints port-Hamiltonian robots

> ”As far as the laws of mathematics refer to reality, they are not certain, and as far as they are certain, they do not refer to reality.”
>
> - Albert Einstein

In this chapter a family of virtual contraction based controllers that solve the standard tracking problem of flexible-joint robots (FJRs) is proposed. The FJRs are modeled as class of underactuated mechanical port-Hamiltonian (pH) systems. Under potential energy matching conditions, the closed-loop virtual system is contractive. It is also shown that the closed-loop virtual system associated to the FJR preserves the structural propertied of the closed-loop fully-actuated virtual system studied in Chapter 4. Moreover, conditions for differential passivity and the existence of (incrementally) passive maps are given. A two flexible joints robot manipulator is taken as case of study for which two tracking schemes of the aforementioned family of controllers are constructed. Two novel tracking schemes of the controllers family will be constructed in detail, and evaluated experimentally on a planar FJR of two degrees of freedom.

5.1 Introduction

Control of rigid robots has been widely studied in the literature since they are instrumental in modern manufacturing systems. However, as reviewed in (Nicosia and Tomei 1995) the elasticity in the joints often can not be neglected for accurate position tracking. For every joint that is actuated by a motor, two degrees of freedom are required instead of only one. Such FJRs are therefore *underactuated* mechanical systems. In the work of (Spong 1987) two state feedback control laws based, respectively, on feedback linearization and on singular perturbation theory are presented for a simplified model. Similarly, in (Canudas de Wit et al. 2012) a dynamic feedback controller for a more detailed model is presented. In (Loria and Ortega 1995) a computed-torque controller for FJRs is designed, which does not need *jerk* measurements. In (Ailon and Ortega 1993) and (Brogliato et al. 1995) passivity-based control (PBC) schemes are proposed. The first one is an observer-based controller which requires only motor position measurements. In the latter one, a PBC controller is designed and compared with backstepping and decoupling techniques. For further details on PBC of FJRs we refer to (Ortega et al. 2013) and references therein. In (Astolfi and Ortega 2003a), a global tracking controller based on the immersion and invariance (I&I) method is introduced. From a practical point of view, in (Albu-Schäffer et al. 2007), a torque feedback is embedded into the passivity-based control approach, leading to a full state feedback controller, where acceleration and jerk measurements are not required. In the recent work of (Avila-Becerril et al. 2016), a dynamic controller is designed which for global position tracking of FJRs based only on measurements of link and joint positions.

All of the above control methods are designed for FJRs modeled as second order Euler-Lagrange (EL) systems. Most of these schemes are based on the selection of a suitable storage function that together with the dissipativity of the closed-loop system, ensures the convergence of the state trajectories to the desired solution.

As an alternative to the EL formalism, the pH framework (van der Schaft and Maschke 1995a) has been considered for the control of JFRs. Some set-point controllers have been proposed for FJRs modeled as pH systems. For instance in (Ortega and Borja 2014b) the controller for FJRs modeled as EL systems in (Ortega et al. 2013) is adapted and interpreted in terms of the Control by Interconnection technique[1] (CbI). In (Zhang et al. 2014), they propose an Interconnection and Damping Assignment PBC (IDA-PBC[2]) scheme, where the controller is designed with respect to the pH representation of the EL-model in (Albu-Schäffer et al. 2007). The latest result on set-point regulation of FJRs in the pH framework is presented in (Borja et al. 2018), where nonlinear passivity based PID controllers are designed by using alternative passive output.

5.2 Flexible-joints robots as port-Hamiltonian systems

FJRs are a class of robot manipulators in which each joint is given by a link interconnected to a motor through a spring. Two generalized coordinates are needed to describe the configuration of a single flexible-joint, these are given by the link q_ℓ and motor q_m positions, see Figure 5.1.

Thus, FJRs are a class of *underactuated* mechanical systems of $n = \dim Q$ degrees of freedom (dof) from which the dof corresponding to the n_m-motors position are actu-

[1] We refer interested readers on CbI to (Ortega et al. 2008).

[2] For IDA-PBC technique see also (Ortega et al. 2002).

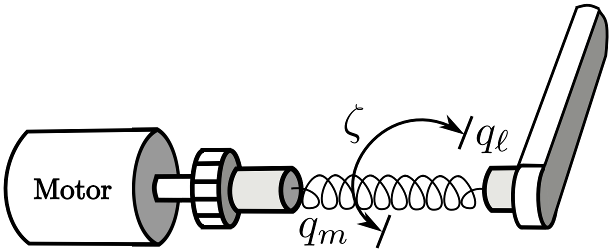

Figure 5.1: Motor's shaft position q_m, spring's deflection ζ and link's position q_ℓ.

ated and the dof corresponding to the $n_\ell = n_m$ links position are underactuated, where $n = n_m + n_\ell$ and Q is the configuration manifold. The following standard modeling assumptions in Spong (Spong 1987) and Jardón-Kojakhmetov(Jardón-Kojakhmetov et al. 2016) are considered:

- The deflection/elongation ζ of each spring is small enough so that it is represented by a linear model.
- The i-th motor driving the i-link is mounted at the $(i-1)$-link.
- Each motor's center of mass is located along the rotation axes.

The FJR's generalized position $q \in Q$ is split into $q = [q_\ell^\top, q_m^\top]^\top \in Q = Q_{n_\ell} \times Q_{n_m}$, the inertia and damping matrices are assumed to be block partitioned as follows

$$M(q) = \begin{bmatrix} M_\ell(q_\ell) & 0_{n_\ell} \\ 0_{n_m} & M_m(q_m) \end{bmatrix}; \qquad D(q) = \begin{bmatrix} D_\ell(q_\ell) & 0_{n_\ell} \\ 0_{n_m} & D_m(q_m) \end{bmatrix}, \tag{5.1}$$

where $M_\ell(q_\ell)$ and $M_m(q_m)$ are the link and motors inertias and $D_\ell(q_\ell)$ and $D_m(q_m)$ are the link and motor damping matrices. The total potential energy is given by

$$P(q) = P_{\ell g}(q_\ell) + P_{mg}(q_m) + P_\zeta(\zeta), \tag{5.2}$$

with links potential energy $P_\ell(q_\ell)$, motors potential energy $P_{mg}(q_m)$ and the potential energy due to the joints stiffness $P_\zeta(\zeta)$. The potential energy for linear springs is

$$P_\zeta(\zeta) = \frac{1}{2}\zeta^\top K\zeta, \tag{5.3}$$

with $\zeta := q_m - q_\ell$ and the stiffness coefficients matrix $K \in \mathbb{R}^{n\times n}$ is symmetric and positive definitive. Since $\text{rank}(B(q)) = n_m$, the input matrix is given as $B(q) = [0_{n_\ell} B_m^\top(q_m)]^\top$. Substitution of the above specifications in the Hamiltonian function (3.22) and the mechanical system (3.24) results in the port-Hamiltonian model for a FJR given by

$$\begin{aligned}\begin{bmatrix}\dot{q}_\ell\\ \dot{q}_m\\ \dot{p}_\ell\\ \dot{p}_m\end{bmatrix} &= \begin{bmatrix}0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m}\\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & I_{n_m}\\ -I_{n_\ell} & 0_{n_m} & -D_\ell & 0_{n_m}\\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & -D_m\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial q_\ell}(q,p)\\ \frac{\partial H}{\partial q_m}(q,p)\\ \frac{\partial H}{\partial p_\ell}(q,p)\\ \frac{\partial H}{\partial p_m}(q,p)\end{bmatrix} + \begin{bmatrix}0_{n_\ell}\\ 0_{n_m}\\ 0_{n_\ell}\\ B_m(q_m)\end{bmatrix} u_m,\\ y &= B_m^\top(q_m)\frac{\partial H}{\partial p_m}(q,p),\end{aligned} \tag{5.4}$$

where $p_\ell = M_\ell(q_\ell)\dot{q}_\ell$ and $p_m = M_m(q_m)\dot{q}_m$ are the links and motors momenta, respectively; and $p = [p_\ell^\top, p_m^\top]^\top$. Without loss of generality we take $B_m(q_m) = I_{n_m}$. The pH-FJR (5.4) can be rewritten as the alternative model (3.26) with

$$E(x) = \begin{bmatrix}S_\ell(q_\ell,\dot{q}_\ell) - \frac{1}{2}\dot{M}_\ell(q_\ell) & 0_{2n_m}\\ 0_{2n_\ell} & S_m(q_m,\dot{q}_m) - \frac{1}{2}\dot{M}_m(q_m)\end{bmatrix}_{\dot{q}=M^{-1}(q)p}, \tag{5.5}$$

with $S_\ell^\top(q_\ell, p_\ell) = -S_\ell(q_\ell, p_\ell)$ and $S_m^\top(q_m, p_m) = -S_m(q_m, p_m)$. The state vector of (5.4) will be denoted by $x := [q^\top, p^\top]^\top \in T^*Q$.

5.3 Trajectory tracking problem for FJRs

The trajectory control problem described in Section 2.2.4 for FJRs reads as follows: Given a desired trajectory $q_{\ell d}(t)$ for the link's position $q_\ell(t)$, design the control law u_m for the pH-FJR (5.4) such that the link's position $q_\ell(t)$ converges asymptotically/exponentially to the desired trajectory $q_{\ell d}(t)$ and all closed-loop system's trajectories are bounded.

Proposed solution: By means of the v-CBC, the tracking problem is solved as proposed in Section 2.2.4 by designing the composite controller given by

$$\zeta(x_v, x, t) := u_v^{ff}(x_v, x, t) + u_v^{fb}(x_v, x, t) \tag{5.6}$$

where the *feedforward-like* term $u_v^{ff}(x_v, x, t)$ ensures that the closed-loop virtual system has the desired trajectory $x_d(t)$ as steady-state solution, and the *feedback* action $u_v^{fb}(x_v, x, t)$ enforces the closed-loop virtual system to be contractive or differentially passive.

5.4 Control design procedure via v-CBC

Remark 5.1. *It will be shown that the controller for the rigid robot developed in Proposition 4.2 can be extended to the case of FJRs. Thus, in order to avoid notational inconsistence between the rigid and flexible controllers, the subscript ℓ is added to the state and parameters in* (4.5)-(4.6)*, i.e.,*

$$u_{\ell v}, \quad x_\ell = [q_\ell^\top, p_\ell^\top], \quad D_\ell(q_\ell), \quad E_\ell(x_\ell), B_\ell(q_\ell).$$

Step 1: Virtual mechanical system for a pH-FJR

Using (5.5), the corresponding virtual system (3.36) for the pH-FJR (5.4) is given by

$$\begin{aligned}\dot{x}_v &= \begin{bmatrix} 0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & I_{n_m} \\ -I_{n_\ell} & 0_{n_m} & -(E_\ell(x_\ell)+D_\ell(q_\ell)) & 0_{n_m} \\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & -(E_m(x_m)+D_m(q_m)) \end{bmatrix} \frac{\partial H_v}{\partial x_v}(q_v, p_v, q) + \begin{bmatrix} 0_{n_\ell} \\ 0_{n_m} \\ 0_{n_\ell} \\ I_{n_m} \end{bmatrix} u_{mv}, \\ y_v &= \begin{bmatrix} 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & I_{n_m} \end{bmatrix}^\top \frac{\partial H_v}{\partial x_v}(q_v, p_v, q). \end{aligned} \tag{5.7}$$

with $H_v(q_v, p_v, q)$ as in (3.32) with respect to (5.1)-(5.3) and $x_v = [q_v^\top, p_v^\top]^\top \in T^*\mathcal{Q}$, with $q_v = [q_{\ell v}^\top, q_{mv}^\top]^\top$ and $p_v = [p_{\ell v}^\top, p_{mv}^\top]^\top$.

Step 2: Contraction-based control design of the virtual FJR

Notice that in the links momentum dynamics of the virtual system (5.7)

$$\dot{p}_{\ell v} = -\frac{\partial P_{\ell v}}{\partial q_{\ell v}}(q_\ell) - [E_\ell(x_\ell) + D_\ell(q_\ell)]\, M_\ell^{-1}(q_\ell) p_{\ell v} + K\zeta_v,$$

the potential force $K\zeta_v = K(q_{mv} - q_{\ell v})$ acts in all the dof since $\text{rank}(K) = n_\ell$. Following the ideas in (Brogliato et al. 1995, Ott et al. 2008) for the passivity approach, it is desired to find a desired motors position reference q_{md} such that the supplied torque by the springs to the links makes the link's position to track a desired time-varying reference $q_{\ell d}(t)$. Accordingly, the following potential forces relation should hold

$$\frac{\partial P_{\zeta v}}{\partial q_{mv}}(q_{\ell v}, q_{mv}) = K(q_{mv} - q_{\ell v}) = \frac{\partial \overline{P}_{\zeta v}}{\partial q_{mv}}(q_m, q_{md}, q_{\ell v}, t) := K(q_m - q_{md}) + u_{\ell v}, \tag{5.8}$$

for any q_{mv} and $q_{\ell v}$, where $u_{\ell v}$ is an artificial input for the links dynamics, $P_{\zeta_v}(\zeta_v)$ is the virtual potential energy following the form in (5.3) and $\overline{P}_{\zeta_v}(\zeta_v)$ is the target virtual potential energy. Solving for q_{md} the matching condition (5.8) yields $q_{md} = q_{\ell v} + K^{-1}u_{\ell v}$.

Proposition 5.2. *Consider the actual system* (5.4) *and its virtual system* (5.7). *Consider also the controller $u_{\ell v}$ in* (4.6). *Let $x_{md} = [q_{md}^\top, p_{md}^\top]^\top \in T^*\mathcal{Q}_m$ be the motor reference state, with $q_{md} = q_{\ell v} + K^{-1}u_{\ell v}$. Let us introduce the motors error coordinates as*

$$\tilde{x}_{mv} := \begin{bmatrix} \tilde{q}_{mv} \\ \sigma_{mv} \end{bmatrix} = \begin{bmatrix} q_{mv} - q_{md} \\ p_{mv} - p_{mr} \end{bmatrix}, \tag{5.9}$$

where the artificial motor momentum reference p_{mr} is defined by

$$p_{mr} := M_m(q_m)(\dot{q}_{md} - \phi_m(\tilde{q}_{mv}) + \overline{v}_{mr}), \tag{5.10}$$

with $\delta\overline{v}_{mr} = -\Pi_m^{-1}(\tilde{q}_{mv}, t)K^\top M_\ell^{-\top}(q_\ell)\sigma_{\ell v}$, and function $\phi_m : \mathcal{Q}_m \to T_{\tilde{q}_{mv}}\mathcal{Q}_m$ and a positive definite Riemannian metric $\Pi_m : \mathcal{Q}_m \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n_m \times n_m}$ satisfying

$$\dot{\Pi}_m(\tilde{q}_{mv}, t) - \Pi_m(\tilde{q}_{mv}, t)\frac{\partial \phi_m}{\partial \tilde{q}_{mv}}(\tilde{q}_{mv}) - \frac{\partial \phi_m^\top}{\partial \tilde{q}_{mv}}(\tilde{q}_{mv})\Pi_m(\tilde{q}_{mv}, t) \leq -2\beta_m(\tilde{q}_{mv}, t)\Pi_m(\tilde{q}_{mv}, t), \tag{5.11}$$

with $\beta_m(\tilde{q}_{mv}, t) > 0$. Suppose that the i-th row of $\Pi_m(\tilde{q}_{mv}, t)$ is a conservative vector field. Then, the virtual system (5.7) *in closed-loop with the control law given by*

$$u_{mv}(x_v, x, t) := u_{mv}^{ff}(x_v, x, t) + u_{mv}^{fb}(x_v, x, t), \tag{5.12}$$

with

$$\begin{aligned} u_{mv}^{ff}(x_v, x, t) &= \dot{p}_{mr} + \frac{\partial P_m}{\partial q_{mv}}(q_{mv}) + k\zeta_v + [E_m(x_m) + D_m(q_m)]M_m^{-1}(q_m)p_{mr}, \\ u_{mv}^{fb}(x_v, x, t) &= -\int_{0_{n_m}}^{\tilde{q}_{mv}} \Pi_m(\xi_{mv}, t)d\xi_{mv} - K_{md}M_m^{-1}(q_m)\sigma_{mv} + \omega_m, \end{aligned} \tag{5.13}$$

is strictly differentially passive from $\delta\omega$ to $\delta y_{\sigma_v} = M^{-1}(q)\delta\sigma_v$ with respect to the differential storage function

$$W(\tilde{x}_v, \delta\tilde{x}_v, t) = \frac{1}{2}\delta\tilde{x}_v^\top \begin{bmatrix} \Pi_{\tilde{q}_v}(\tilde{q}_v, t) & 0_n \\ 0_n & M^{-1}(q) \end{bmatrix} \delta\tilde{x}_v, \tag{5.14}$$

where the error coordinate is $\tilde{x}_v = [\tilde{q}_v^\top, \sigma_v^\top]^\top$ with $\tilde{q}_v := [\tilde{q}_{\ell v}^\top, \tilde{q}_{mv}^\top]^\top$ and $\sigma_v := [\sigma_{\ell v}^\top, \sigma_{mv}^\top]^\top$. The derivative matrix gain K_{md} is constant and positive definite, $\omega = [\omega_\ell^\top, \omega_m^\top]^\top$ is an external input and $\Pi_{\tilde{q}_v}(\tilde{q}_v, t) := diag\{\Pi_\ell(\tilde{q}_{\ell v}, t), \Pi_m(\tilde{q}_{mv}, t)\}$.

Proof: First introduce, for sake of clarity, the following variables

$$\begin{aligned} p_r &:= [p_{\ell r}^\top, p_{mr}^\top]^\top, \bar{v}_r := [\bar{v}_{\ell r}^\top, \bar{v}_{mr}^\top]^\top, q_d := [q_{\ell d}, q_{md}]^\top, \\ \phi &:= [\phi_\ell^\top, \phi_m^\top]^\top, K_d := \mathrm{diag}\{K_{d\ell}, K_{dm}\}. \end{aligned} \tag{5.15}$$

Consider the dynamics of q_v in (5.7) with $y_{\tilde{q}_v} = q_v - q_d$ as output and p_v as "input". Define the control-like input as $p_v = \sigma_v + p_r$, where σ_v is a new input and p_r as defined in (4.3) and (5.10), respectively. Then, the "closed-loop" position dynamics is

$$\begin{cases} \dot{q}_v = M^{-1}(q_d)p_d - \phi(q_v - q_d) + M^{-1}(q)\sigma_v + \bar{v}_r, \\ y_{\tilde{q}_v} = q_v - q_d, \end{cases} \tag{5.16}$$

whose associated prolonged system (see (2.10)), in coordinates (4.2)-(5.9), is given by

$$\tilde{\Sigma}_{\sigma_v}^\delta : \begin{cases} \dot{\tilde{q}}_v = -\phi(\tilde{q}_v) + M^{-1}(q)\sigma_v + \bar{v}_r \\ \delta\dot{\tilde{q}}_v = -\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v)\delta\tilde{q}_v + M^{-1}(q)\delta\sigma_v + \delta\bar{v}_r \\ y_{\tilde{q}_v} = \tilde{q}_v, \\ \delta y_{\tilde{q}_v} = \delta\tilde{q}_v. \end{cases} \tag{5.17}$$

Now, take as candidate differential Lyapunov function to

$$W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) = \frac{1}{2}\delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\delta\tilde{q}_v. \tag{5.18}$$

Then, the time derivative of (5.18) along solutions of $\tilde{\Sigma}_\sigma^\delta$ in (5.17),

$$\begin{aligned} \dot{W}_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) &= -\delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v)\delta\tilde{q}_v + \delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)M^{-1}(q)\delta\sigma_v \\ &\quad + \frac{1}{2}\delta\tilde{q}_v^\top \dot{\Pi}_{\tilde{q}_v}(\tilde{q}_v, t)\delta\tilde{q}_v + \delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\delta\bar{v}_r, \\ &= \frac{1}{2}\delta\tilde{q}_v^\top \left[\dot{\Pi}_{\tilde{q}_v}(\tilde{q}_v, t) - \Pi_{\tilde{q}_v}(\tilde{q}_v, t)\frac{\partial\phi}{\partial\tilde{q}_v}(\tilde{q}_v) - \frac{\partial\phi^\top}{\partial\tilde{q}_v}(\tilde{q}_v)\Pi_{\tilde{q}_v}(\tilde{q}_v, t)\right]\delta\tilde{q}_v \\ &\quad + \delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)M^{-1}(q)\left(\delta\sigma_v + M(q)\delta\bar{v}_r\right), \end{aligned} \tag{5.19}$$

Let us take $\delta y_{\tilde{q}_v}^\top := \delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)M^{-1}(q)$. By hypotheses (4.4) and (5.11) it follows that

$$\dot{W}_{\tilde{q}_v} \le -2\underbrace{\min\{\beta_\ell(\tilde{q}_{\ell v}, t), \beta_m(\tilde{q}_{mv}, t)\}}_{=:\beta_{\tilde{q}}(\tilde{q}_v, t)} W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) + \delta y_{\tilde{q}_v}^\top\left(\delta\sigma_v + M(q)\delta\bar{v}_r\right). \tag{5.20}$$

Then, system (5.16) defines the strictly differentially passive map $(\delta\sigma_v + M(q)\delta\bar{v}_r) \mapsto \delta y_{\tilde{q}_v}$. This implies contraction with rate $2\beta(\tilde{q}_v, t)$ when $\delta\sigma_v = 0_n$ due to $\bar{v}_r$ in (5.15). Furthermore, exponential convergence to the equilibrium $\tilde{q}_v = 0_n$ if $\sigma_v = 0_n$ is guaranteed.

We follow a similar procedure for the complete system (5.7) with output $y_{\sigma_v} := p_v - p_r$. System (5.7) in coordinates (4.3) and (5.10) is the system composed (5.17) and

$$\dot{\sigma}_v = \begin{bmatrix} -\frac{\partial P_\ell}{\partial q_{\ell v}}(\tilde{q}_{\ell v} + q_{\ell d}) + K\zeta_v - [E_\ell(x_\ell) + D_\ell(q_\ell)]\, M_\ell^{-1}(q_\ell)[\sigma_{\ell v} + p_{\ell r}] - \dot{p}_{\ell r} \\ -\frac{\partial P_m}{\partial q_{mv}}(\tilde{q}_{mv} + q_{md}) - K\zeta_v - [E_m(x_m) + D_m(q_m)]\, M_m^{-1}(q_m)[\sigma_{mv} + p_{mr}] + u_{mv} - \dot{p}_{mr} \end{bmatrix}. \tag{5.21}$$

From the potential energy matching condition (5.8) , i.e., $K\zeta_v = K\tilde{q}_{mv} + u_{\ell v}$, we have that

$$\dot{\sigma}_v = \begin{bmatrix} -\frac{\partial P_\ell}{\partial q_{\ell v}} - [E_\ell(x_\ell) + D_\ell(q_\ell)]\, M_\ell^{-1}(q_\ell)[\sigma_{\ell v} + p_{\ell r}] + u_{\ell v} - \dot{p}_{\ell r} + K\tilde{q}_{mv} \\ -\frac{\partial P_m}{\partial q_{mv}} - (K\tilde{q}_{mv} + u_{\ell v}) - [E_m(x_m) + D_m(q_m)]\, M_m^{-1}(q_m)[\sigma_{mv} + p_{mr}] + u_{mv} - \dot{p}_{mr} \end{bmatrix}. \tag{5.22}$$

Substitution of the feedforward-like actions $u_{\ell v}^{ff}(x_{\ell v}, x_\ell, t)$ and $u_{mv}^{ff}(x_{mv}, x_m, t)$ of (4.5) and (5.12), respectively, in (5.22) gives

$$\dot{\sigma}_v = -[E(x) + D(q)]\, M^{-1}(q)\sigma_v + \begin{bmatrix} u_{\ell v}^{fb} + K\tilde{q}_{mv} \\ u_{mv}^{fb} \end{bmatrix}. \tag{5.23}$$

Notice that above substitution lets us to impose $\sigma_v = 0_n$ as a particular solution of (5.23) when $u_{\ell v}^{fb} = 0_{n_\ell}$ and $u_{mv}^{fb} = 0_{n_m}$. The prolongation of (5.7) after above substitutions; in coordinates (4.3) and (5.10), respectively; is the system formed by $\tilde{\Sigma}_{\sigma_v}^{\delta}$ in (4.9) and

$$\tilde{\Sigma}_{u_{fbv}}^{\delta} : \begin{cases} \dot{\sigma}_v = -[E(x) + D(q)]\, M^{-1}(q)\sigma_v + \begin{bmatrix} u_{\ell v}^{fb} \\ u_{mv}^{fb} \end{bmatrix} + \begin{bmatrix} K\tilde{q}_{mv} \\ 0_{n_m} \end{bmatrix}, \\ \delta\dot{\sigma}_v = -[E(x) + D(q)]\, M^{-1}(q)\delta\sigma_v + \begin{bmatrix} \delta u_{\ell v}^{fb} \\ \delta u_{mv}^{fb} \end{bmatrix} + \begin{bmatrix} K\delta\tilde{q}_{mv} \\ 0_{n_m} \end{bmatrix}, \\ y_{\sigma_v} = \sigma_v, \\ \delta y_{\sigma_v} = \delta\sigma_v, \end{cases} \tag{5.24}$$

Let (5.14) be the differential Lyapunov function for system $\tilde{\Sigma}_{\sigma_v}^{\delta}$-$\tilde{\Sigma}_{u_{fbv}}^{\delta}$. Then

$$\begin{aligned} \dot{W} \leq & -2\beta_{\tilde{q}}(\tilde{q}_v, t)W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) + \delta\overline{y}_{\tilde{q}_v}^{\top}(\delta\sigma_v + M(q)\delta\overline{v}_r) + \frac{1}{2}\delta\sigma_v^{\top}\dot{M}^{-1}(q)\delta\sigma_v \\ & + \delta\sigma_v^{\top} M^{-1}(q)\left(-[E(x) + D(q)]\, M^{-1}(q)\delta\sigma_v + \begin{bmatrix} \delta u_{\ell v}^{fb} \\ \delta u_{mv}^{fb} \end{bmatrix} + \begin{bmatrix} K\delta\tilde{q}_{mv} \\ 0_{n_m} \end{bmatrix}\right), \end{aligned}$$

$$
\begin{aligned}
&= -2\beta_{\tilde{q}}(\tilde{q}_v, t) W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) - \delta\tilde{q}_{mv}^\top K^\top M_\ell^{-\top}(q_\ell)\delta\sigma_{\ell v} \\
&\quad - \delta\sigma_v^\top M^{-1}(q)(D(q) + K_d)M^{-1}(q)\delta\sigma_v \\
&\quad + \delta\sigma_v^\top M^{-1}(q)\delta\omega + \delta\sigma_{\ell v}^\top M_\ell^{-1}(q) K \delta\tilde{q}_{mv}, \\
&\leq -2\beta_{\tilde{q}}(\tilde{q}_v, t) W_{\tilde{q}_v}(\tilde{q}_v, \delta\tilde{q}_v, t) \\
&\quad - \lambda_{\min}\{D(q) + K_d\}\lambda_{\min}\{M^{-1}(q)\}\delta\sigma_v^\top M^{-1}(q)\delta\sigma_v + \delta\sigma_v^\top M^{-1}(q)\delta\omega, \\
&\leq -2\min\{\beta_{\tilde{q}}(\tilde{q}_v, t), \lambda_{\min}\{D + K_d\}\lambda_{\min}\{M^{-1}(q)\}\} W + \delta y_{\sigma_v}^\top \delta\omega.
\end{aligned}
\tag{5.25}
$$

The step from the first to the second inequality follows after substitution of the variational controllers $\delta u_{\ell v}^{fb}$ and δu_{mv}^{fb} from (4.6) and (5.13), respectively; and using identity

$$
\dot{M}(q) = -M(q)\dot{M}^{-1}(q)M(q),
$$

together with the definition of matrices $E_\ell(x_\ell)$ and $E_m(x_m)$ in (5.5). From the second to the third inequality, the differentially passive output $\delta\overline{y}_{\tilde{q}_v}^\top := \delta\tilde{q}_v^\top \Pi_{\tilde{q}_v}(\tilde{q}_v, t)M^{-1}(q)$ has been substituted. From the third to the fourth inequality, the crossing terms have canceled and it has been considered that $\overline{v}_{\ell r} = 0_{n_\ell}$. The proof is then completed by using the lower bounds of matrices $D(q)$ and $M^{-1}(q)$. ∎

Step 3: Original pH-FJRs tracking controller

Notice that (5.25) implies contracting behavior of the virtual system (5.7) in closed-loop with (5.12) for $\delta\omega = 0_n$. Furthermore, the origin $(\tilde{q}_v, \sigma_v) = (0_n, 0_n)$ is a solution of the closed-loop system if $\omega = 0_n$. Using these facts, in the next result we propose a family of trajectory-tracking controllers for the pH-FJR (5.4).

Corollary 5.3. *Consider the virtual controller* (5.12) *and let $q_{\ell d}(t) \in \mathcal{Q}_\ell$ be a desired time-varying trajectory. Suppose that the FJR* (5.4) *is controlled by the scheme*

$$
u_m(x, t) := u_{mv}(x, x, t). \tag{5.26}
$$

Then closed-loop system's links position q_ℓ converges exponentially to the trajectory $q_d(t)$ with rate

$$
\beta = 2\min\{\beta_{\tilde{q}}(\tilde{q}_v, t), \lambda_{\min}\{D(q) + K_d\}\lambda_{\min}\{M^{-1}(q)\}\}c_1 = \min\{\lambda_{\min}(\Pi_{\tilde{q}_v}), \lambda_{\min}(M^{-1})\}. \tag{5.27}
$$

Proof: The proof follows similar to that for Corollary 4.4. System (5.4) in closed-loop with (5.26) defines a solution of the virtual system (5.7) in closed-loop with (5.12) whenever $x_v = x$. With $\omega = 0_n$ in (5.25) and the differential storage function (4.7) qualifies as differential Lyapunov function satisfying

$$W(\tilde{x}_v, \delta\tilde{x}_v, t) < e^{-2\beta t} W(\tilde{x}_v(0), \delta\tilde{x}_v(0), t), \quad \text{for all} \quad t.$$

Since the origin $\tilde{x}_v = 0_{2n}$ is also a solution, above implies that the Finsler's distance (2.13) with structure $\mathcal{F}(x_v, \delta x_v, t) = \sqrt{W(x_v, \delta x_v, t)}$ between solutions $\tilde{x}_v = 0_{2n}$ and $\tilde{x}_v = \tilde{x}$ shrinks, i.e.,

$$d(\tilde{x}, 0_n) < e^{-\beta t} W(\tilde{x}_v(0), \delta\tilde{x}_v(0), t), \quad \forall t > t_0. \tag{5.28}$$

Therefore, the pH-FJR's links position q_ℓ converges exponentially to $q_{\ell d}$ as $t \to \infty$ with rate β. Moreover, since $C = T^*Q$ is the contraction region for the closed-loop system, the convergence property is global. ■

5.5 Properties of the closed-loop virtual system

5.5.1 Structural properties

In the following result we show that system (5.7) in closed-loop with (5.12) preserves the structure of the variational dynamics (2.24).

Corollary 5.4. *Consider system* (5.7) *in closed-loop with* (5.12)*. Then the closed-loop variational dynamics satisfies Lemma 2.13, in coordinates $\tilde{x}_v$, with*

$$\Pi(\tilde{x}_v, t) = \begin{bmatrix} \Pi_\ell(\tilde{q}_{\ell v}, t) & 0_{n_m} & 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & \Pi_m(\tilde{q}_{\ell v}, t) & 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & M_\ell^{-1}(q_\ell) & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & M_m^{-1}(q_m) \end{bmatrix};$$

$$\Xi(\tilde{x}_v, t) = \begin{bmatrix} 0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & -\Pi_m^{-1}(\tilde{q}_{mv}, t)K^\top & I_{n_m} \\ -I_{n_\ell} & K\Pi_m^{-1}(\tilde{q}_{mv}, t) & -S_\ell(q_\ell, p_\ell) & 0_{n_m} \\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & -S_m(q_m, p_m) \end{bmatrix};$$

$$\Upsilon(\tilde{x}_v, t) = \begin{bmatrix} \frac{\partial \phi_\ell}{\partial \tilde{q}_{\ell v}} \Pi_\ell^{-1}(\tilde{q}_{\ell v}, t) & 0_{n_m} & 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & \frac{\partial \phi_m}{\partial \tilde{q}_{mv}} \Pi_m^{-1}(\tilde{q}_{mv}, t) & 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & \left(D_\ell + K_{\ell d} - \frac{1}{2}\dot{M}_\ell\right) & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & \left(D_m + K_{md} - \frac{1}{2}\dot{M}_m\right) \end{bmatrix};$$

$$\Psi = \begin{bmatrix} 0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & I_{n_m} \end{bmatrix}^\top, \tag{5.29}$$

and $\Theta(x_v, t)$ is given by the Jacobian matrix of $\tilde{x}_v = x_v - x_d$, with respect to x_v, where $x_d := [q_{\ell d}^\top, q_{md}^\top, p_{\ell r}^\top, p_{mr}^\top]^\top$ is desired state.

Proof: It will be shown that the closed-loop variational dynamics has the proposed structure (2.24) which satisfies condition (2.25). Consider the closed-loop system

$$\begin{bmatrix} \dot{\tilde{q}}_{\ell v} \\ \dot{\tilde{q}}_{mv} \\ \dot{\sigma}_{\ell v} \\ \dot{\sigma}_{mv} \end{bmatrix} = \begin{bmatrix} -\phi_\ell(\tilde{q}_{\ell v}) + M_\ell^{-1}(q_\ell)\sigma_{\ell v} \\ -\phi_m(\tilde{q}_{mv}) + M_m^{-1}(q_m)\sigma_{mv} + \overline{v}_{mr} \\ -\int_{0_{n_\ell}}^{\tilde{q}_{\ell v}} \Pi_\ell(\xi_\ell, t)d\xi_\ell - [E_\ell(x_\ell) + D_\ell(q_\ell) + K_{\ell d}]\, M_\ell^{-1}(q_\ell)\sigma_{\ell v} + K\tilde{q}_{mv} + \omega_\ell \\ -\int_{0_{n_m}}^{\tilde{q}_{mv}} \Pi_m(\xi_m, t)d\xi_m - [E_m(x_m) + D_m(q_m) + K_{md}]\, M_m^{-1}(q_m)\sigma_{mv} + \omega_m \end{bmatrix}, \tag{5.30}$$

whose variational dynamics is

$$\begin{bmatrix} \delta\dot{\tilde{q}}_{\ell v} \\ \delta\dot{\tilde{q}}_{mv} \\ \delta\dot{\sigma}_{\ell v} \\ \delta\dot{\sigma}_{mv} \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{\partial\phi_\ell}{\partial\tilde{q}_{\ell v}} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & -\frac{\partial\phi_m}{\partial\tilde{q}_{mv}} & -\Pi_m^{-1}(\tilde{q}_{mv}, t)K^\top & I_{n_m} \\ -\Pi_\ell & K & -(E_\ell + D_\ell + K_{\ell d}) & 0_{n_m} \\ 0_{n_\ell} & -\Pi_m & 0_{n_\ell} & -(E_m + D_m + K_{md}) \end{bmatrix}}_{\mathcal{A}(\tilde{x}_v, t)} \begin{bmatrix} \delta\tilde{q}_{\ell v} \\ \delta\tilde{q}_{mv} \\ M_\ell^{-1}\delta\sigma_{\ell v} \\ M_m^{-1}\delta\sigma_{mv} \end{bmatrix} \tag{5.31}$$

$$+ \begin{bmatrix} 0_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & I_{n_m} \end{bmatrix}^\top \begin{bmatrix} \delta\omega_\ell \\ \delta\omega_m \end{bmatrix}.$$

Factorizing $\Pi_\ell(\tilde{q}_{\ell v}, t)$ and $\Pi_m(\tilde{q}_{mv}, t)$ from the first, respectively second columns of $\mathcal{A}(\tilde{x}_v, t)$ in (5.31) results in

$$\begin{bmatrix} \delta\dot{\tilde{q}}_{\ell v} \\ \delta\dot{\tilde{q}}_{mv} \\ \delta\dot{\sigma}_{\ell v} \\ \delta\dot{\sigma}_{mv} \end{bmatrix} = \begin{bmatrix} -\frac{\partial\phi_\ell}{\partial\tilde{q}_{\ell v}}\Pi_\ell^{-1} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & -\frac{\partial\phi_m}{\partial\tilde{q}_{mv}}\Pi_m^{-1} & -\Pi_m^{-1}K^\top & I_{n_m} \\ -I_{n_\ell} & K\Pi_m^{-1} & -A_\ell & 0_{n_m} \\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & -A_m \end{bmatrix} \begin{bmatrix} \Pi_\ell\delta\tilde{q}_{\ell v} \\ \Pi_m\delta\tilde{q}_{mv} \\ M_\ell^{-1}\delta\sigma_{\ell v} \\ M_m^{-1}\delta\sigma_{mv} \end{bmatrix} + \Psi\delta\omega. \tag{5.32}$$

$$\delta\dot{\tilde{x}}_v = [\Xi(\tilde{x}_v, t) - \Upsilon(\tilde{x}_v, t)]\,\Pi(\tilde{x}_v, t)\delta\tilde{x}_v + \Psi\delta\omega.$$

where $A_\ell := (E_\ell + D_\ell + K_{\ell d})$ and $A_m := (E_m + D_m + K_{md})$. Indeed, (5.32) is of the form (2.24). This proves implicitly that the differential transformation $\Theta(x_v, t)$ exists.

Now, condition (2.25) in terms of $\Pi(\tilde{x}_v, t)$ and $\Upsilon(\tilde{x}_v, t)$ in (5.29) is explicitly given by

$$\begin{bmatrix} \delta\tilde{q}_{\ell v} \\ \delta\tilde{q}_{mv} \\ \delta\sigma_{\ell v} \\ \delta\sigma_{mv} \end{bmatrix}^\top \begin{bmatrix} Y_{11} & 0_{n_m} & 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & Y_{22} & 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & -M_\ell^{-1}(D_\ell + K_{\ell d})M_\ell^{-1} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} & 0_{n_\ell} & -M_m^{-1}(D_m + K_{md})M_m^{-1} \end{bmatrix} \begin{bmatrix} \delta\tilde{q}_{\ell v} \\ \delta\tilde{q}_{mv} \\ \delta\sigma_{\ell v} \\ \delta\sigma_{mv} \end{bmatrix}, \tag{5.33}$$

with

$$Y_{11} = \dot{\Pi}_\ell - \Pi_\ell \frac{\partial \phi_\ell}{\partial \tilde{q}_\ell} - \frac{\partial \phi_\ell^\top}{\partial \tilde{q}_\ell} \Pi_\ell, \quad \text{and} \quad Y_{22} = \dot{\Pi}_m - \Pi_m \frac{\partial \phi_m}{\partial \tilde{q}_m} - \frac{\partial \phi_m^\top}{\partial \tilde{q}_m} \Pi_m.$$

It is straightforward to verify that (5.33) coincides with the time derivative of (5.14) for $\delta\omega = 0_n$. Hence condition (2.25) is satisfied. ■

In other words, the statement in Corollary 5.4 means that the differential transformation $\Theta(x_v, t)$ is implicitly constructed via the design procedure of Proposition 5.2. Furthermore, notice that the closed-loop dynamics of both $\sigma_{\ell v}$ and σ_{mv} in (5.30) are actuated by ω_ℓ and ω_m, respectively. In fact, this is a direct consequence of the potential energy matching condition (5.8), making possible to rewrite the error dynamics as a "fully-actuated" like system in (5.22). With this interpretation of the closed-loop system (5.30) the structural properties of the v-CBC scheme for fully-actuated systems presented in Section 4.2 are extended to the FJRs case.

Corollary 5.5. *Consider system* (5.7) *in closed-loop with* (5.12). *Assume that the Jacobian matrices* $\frac{\partial \phi_\ell}{\partial \tilde{q}_\ell}(\tilde{q}_{\ell v})$ *and* $\frac{\partial \phi_m}{\partial \tilde{q}_m}(\tilde{q}_{mv})$ *are symmetric and assume that the matrix products* $\Pi_\ell(\tilde{q}_{\ell v}, t)\frac{\partial \phi_\ell}{\partial \tilde{q}_\ell}(\tilde{q}_{\ell v})$ *and* $\Pi_m(\tilde{q}_{mv}, t)\frac{\partial \phi_m}{\partial \tilde{q}_m}(\tilde{q}_{mv})$ *commute. Then the closed-loop variational system preserves the structure of* (3.42), *in coordinates* $\tilde{x}_v$, *with*

$$\frac{\partial^2 \tilde{H}_v}{\partial x_v^2}(\tilde{x}_v, x) = \Pi(\tilde{x}_v, t) \quad \tilde{J}_v(\tilde{x}_v, t) = \Xi(\tilde{x}_v, t), \quad \tilde{R}_v(\tilde{x}_v, t) = \Upsilon(\tilde{x}_v, t), \quad \tilde{g} := \Psi^\top. \tag{5.34}$$

Proof: By Corollary 5.4 it is clear that $\Xi(\tilde{x}_v, t) = -\Xi^\top(\tilde{x}_v, t)$. Under the above hypotheses the products $\Pi_\ell(\tilde{q}_{\ell v}, t)\frac{\partial \phi_\ell}{\partial \tilde{q}_\ell}(\tilde{q}_{\ell v})$ and $\Pi_m(\tilde{q}_{mv}, t), \frac{\partial \phi_m}{\partial \tilde{q}_m}(\tilde{q}_{mv})$ are symmetric. Hence, with $\Upsilon(\tilde{x}_v, t)$ defined as in (5.29), we have that $\Upsilon(\tilde{x}_v, t) = \Upsilon^\top(\tilde{x}_v, t)$. ■

Notice that all matrices in (5.34), which define the variational system in Corollary 5.5, are state and time dependent, while the ones of the variational system (3.42) are only time dependent; in this sense the system in Corollary 5.5 is more general. However, despite of the structure of the variational dynamics (3.42) is preserved, the system defined by (5.34) does not necessarily correspond to a pH-like mechanical system as in (3.40). This would be the case under the following if and only if conditions:

$$\Pi_\ell(\tilde{q}_{\ell v}, t) = \frac{\partial \phi_\ell}{\partial \tilde{q}_{\ell v}}(\tilde{q}_{\ell v}) \quad \text{and} \quad \Pi_m(\tilde{q}_{mv}, t) = \frac{\partial \phi_m}{\partial \tilde{q}_{mv}}(\tilde{q}_{mv}) = \Lambda_m \tag{5.35}$$

where Λ_m is a constant symmetric and positive definite matrix. Indeed, these are necessary and sufficient conditions under which Corollary 4.5 is extended to the FJRs case.

Substitution in the closed-loop system (5.30) gives

$$
\begin{aligned}
\dot{\tilde{x}}_v &= \begin{bmatrix} -I_{n_\ell} & 0_{n_m} & I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & -I_{n_m} & -\Lambda_m^{-1}K^\top & I_{n_m} \\ -I_{n_\ell} & K\Lambda_m^{-1} & -(E_\ell + D_\ell + K_{\ell d}) & 0_{n_m} \\ 0_{n_\ell} & -I_{n_m} & 0_{n_\ell} & -(E_m + D_m + K_{md}) \end{bmatrix} \frac{\partial \tilde{H}_v}{\partial \tilde{x}_v}(\tilde{x}_v, x) + \begin{bmatrix} 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} \\ I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & I_{n_m} \end{bmatrix} \omega, \\
\tilde{y}_v &= \begin{bmatrix} 0_{n_\ell} & 0_{n_\ell} & I_{n_\ell} & 0_{n_\ell} \\ 0_{n_m} & 0_{n_m} & 0_{n_m} & I_{n_m} \end{bmatrix} \frac{\partial \tilde{H}_v}{\partial \tilde{x}_v}(\tilde{x}_v, x),
\end{aligned}
\tag{5.36}
$$

where the x-parametrized closed-loop error Hamiltonian function is given by

$$
\tilde{H}_v(\tilde{x}_v, x) = \frac{1}{2}\tilde{x}_v^\top \Pi(x)\tilde{x}_v = \int_{0_{n_\ell}}^{\tilde{q}_{\ell v}} \phi_\ell(\overline{q}_{\ell v})d\overline{q}_{\ell v} + \frac{1}{2}\tilde{q}_{mv}^\top \Lambda_m \tilde{q}_{mv} + \frac{1}{2}\sigma_v^\top M^{-1}(q)\sigma_v. \tag{5.37}
$$

It has been shown in Corollary 4.5 that for the v-CBC of fully-actuated mechanical systems, the structure in (3.40) is preserved for the tensors $\Pi_{\tilde{q}_v}(\tilde{q}_v, t)$ satisfying $\Pi_{\tilde{q}_v}(\tilde{q}_v, t) = \frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v)$, rather than the more restrictive condition (5.35). However, in this case for pH-FJRs, condition (5.35) is necessary and sufficient. This can be easily seen by noticing that the interconnection matrix $\tilde{J}_v(\tilde{x}_v, t) = \Xi(\tilde{x}_v, t)$ defined in (5.29) contains the term $K\Pi_m^{-1}(\tilde{q}_{mv}, t)$ which due to condition $\Pi_m(\tilde{q}_v, t) = \frac{\partial \phi_m}{\partial \tilde{q}_{mv}}(\tilde{q}_{mv})$ implies that $\Pi_m^{-1}(\tilde{q}_{mv}, t)$ must be either a function only of $\tilde{q}_{mv}$ or a constant matrix. The only possibility is to have $\Pi_m(\tilde{q}_{mv}, t)$ as a constant; otherwise the structure obtained from integrating the variational system (5.32) does not correspond to the pH-like system (3.40). On the other hand, if we assume that the structure (3.40) is preserved, the only way that the variational system corresponds to (5.32) is when $\Pi_m(\tilde{q}_{mv}, t)$ is taken to be constant.

5.5.2 Differential passivity properties

A differential passivity interpretation of system (5.7) in closed-loop with the scheme (5.12) is given. Before stating the result, from (5.31), the closed-loop variational system of the links error state $\tilde{x}_{\ell v}$ is presened[3]

$$
\begin{bmatrix} \delta\dot{\tilde{q}}_{\ell v} \\ \delta\dot{\sigma}_{\ell v} \end{bmatrix} = \begin{bmatrix} -\frac{\partial \phi_\ell}{\partial \tilde{q}_{\ell v}}\Pi_\ell^{-1} & I_{n_\ell} \\ -I_{n_\ell} & -(E_\ell + D_\ell + K_{\ell d}) \end{bmatrix} \begin{bmatrix} \Pi_\ell \delta\tilde{q}_{\ell v} \\ M_\ell^{-1}\delta\sigma_{\ell v} \end{bmatrix} + \begin{bmatrix} I_{n_\ell} & 0_{n_\ell} \\ 0_{n_\ell} & I_{n_\ell} \end{bmatrix} \begin{bmatrix} \delta\overline{v}_{\ell r} \\ K\delta\tilde{q}_{md} + \delta\omega_\ell \end{bmatrix}
\tag{5.38}
$$

[3] For sake of presentation, we explicitly consider the two components of vector $\overline{v}_r = [\overline{v}_{\ell r}^\top, \overline{v}_{mr}^\top]^\top$ in (5.15), even though we know in advance that $\overline{v}_{\ell r} = 0_{n_\ell}$.

which by Lemma 5.4, preserves the structure of (2.24) and is given by

$$\begin{aligned}
\delta\tilde{x}_{\ell v} &= \underbrace{\begin{bmatrix} -\frac{\partial\phi_\ell}{\partial\tilde{q}_{\ell v}}\Pi_\ell^{-1} & I_{n_\ell} \\ -I_{n_\ell} & -(E_\ell + D_\ell + K_{\ell d}) \end{bmatrix}}_{\Xi_\ell(\tilde{x}_\ell,t)-\Upsilon_\ell(\tilde{x}_\ell,t)} \frac{\partial^2\tilde{H}_\ell}{\partial\tilde{x}_{\ell v}^2}(\tilde{x}_{\ell v}, x_\ell, t)\delta\tilde{x}_{\ell v} + \begin{bmatrix} I_{n_\ell} & 0_{n_\ell} \\ 0_{n_\ell} & I_{n_\ell} \end{bmatrix}\begin{bmatrix} \delta\overline{v}_{\ell r} \\ \delta\overline{\omega}_\ell \end{bmatrix}, \\
\delta\tilde{y}_\ell &= \begin{bmatrix} I_{n_\ell} & 0_{n_\ell} \\ 0_{n_\ell} & I_{n_\ell} \end{bmatrix} \frac{\partial^2\tilde{H}_\ell}{\partial\tilde{x}_{\ell v}^2}(\tilde{x}_{\ell v}, x, t)\delta\tilde{x}_{\ell v},
\end{aligned} \tag{5.39}$$

where the differential input $\delta\overline{\omega}_\ell = (K\delta\tilde{q}_{md} + \delta\omega_\ell)$ and the metric tensor of (2.24) is given by the *Hessian* of the energy-like function (see Definition 2.12)

$$\tilde{H}_\ell(\tilde{x}_{\ell v}, x_\ell, t) := \frac{1}{2}\tilde{x}_{\ell v}^\top \begin{bmatrix} \Pi_\ell^{-1}(\tilde{q}_{\ell v}, t) & 0_{n_\ell} \\ 0_{n_\ell} & M_\ell^{-1}(q_\ell) \end{bmatrix} \tilde{x}_{\ell v}. \tag{5.40}$$

Moreover, the map $\begin{bmatrix} \delta\overline{v}_{\ell r}^\top & \delta\omega_\ell^\top \end{bmatrix}^\top \mapsto \delta\tilde{y}_\ell$ is strictly differentially passive with respect to the differential storage function

$$W_\ell(\tilde{x}_{\ell v}, \delta\tilde{x}_{\ell v}, t) = \frac{1}{2}\delta\tilde{x}_{\ell v}^\top \frac{\partial^2\tilde{H}_\ell}{\partial\tilde{x}_{\ell v}^2}(\tilde{x}_{\ell v}, x_\ell, t)\delta\tilde{x}_{\ell v}. \tag{5.41}$$

Similarly, from (4.18), the variational dynamics of the motor error state $\tilde{x}_{mv}$ is

$$\begin{aligned}
\delta\tilde{x}_{mv} &= \begin{bmatrix} -\frac{\partial\phi_m}{\partial\tilde{q}_{mv}}\Pi_m^{-1} & I_{n_m} \\ -I_{n_m} & -(E_m + D_m + K_{md}) \end{bmatrix} \frac{\partial^2\tilde{H}_m}{\partial\tilde{x}_{mv}^2}(\tilde{x}_{mv}, x_m, t)\delta\tilde{x}_{mv} + \begin{bmatrix} I_{n_m} & 0_{n_m} \\ 0_{n_m} & I_{n_m} \end{bmatrix}\begin{bmatrix} \delta\overline{v}_{mr} \\ \delta\overline{\omega}_m \end{bmatrix} \\
\delta\tilde{y}_m &= \begin{bmatrix} I_{n_\ell} & 0_{n_\ell} \\ 0_{n_\ell} & I_{n_\ell} \end{bmatrix} \frac{\partial^2\tilde{H}_m}{\partial\tilde{x}_{mv}^2}(\tilde{x}_{mv}, x_m, t)\delta\tilde{x}_{mv}
\end{aligned} \tag{5.42}$$

with $\delta\overline{v}_{mr} = \Pi_m(\tilde{q}_m, t)K^\top M_\ell^{-1}(q_\ell)\delta\sigma_{\ell v}$ (as stated in (5.15)), $\delta\omega_m = \delta\overline{\omega}_m$ and

$$\tilde{H}_m(\tilde{x}_{mv}, x_m, t) := \frac{1}{2}\tilde{x}_{mv}^\top \begin{bmatrix} \Pi_m^{-1}(\tilde{q}_{mv}, t) & 0_{n_m} \\ 0_{n_m} & M_m^{-1}(q_m) \end{bmatrix} \tilde{x}_{mv}. \tag{5.43}$$

The map $\begin{bmatrix} \delta\overline{v}_{mr}^\top & \delta\omega_m^\top \end{bmatrix}^\top \mapsto \delta\tilde{y}_m$ is strictly differentially passive with respect to the differential storage function

$$W_m(\tilde{x}_{mv}, \delta\tilde{x}_{mv}, t) = \frac{1}{2}\delta\tilde{x}_{mv}^\top \frac{\partial^2\tilde{H}_m}{\partial\tilde{x}_{mv}^2}(\tilde{x}_{mv}, x_m, t)\delta\tilde{x}_{mv}. \tag{5.44}$$

The above observations show that the closed-loop links and motors subsystems are strictly differentially passive. The following proposition shows that the interconnection of these subsystems is also differentially passive.

Corollary 5.6. *Consider the closed-loop links and motors systems and their variational dynamics in* (5.39) *and* (5.44)*, respectively. Then the interconnection of these systems via*

$$\begin{bmatrix} \delta\overline{v}_{\ell r} \\ \delta\overline{\omega}_{\ell} \\ \delta\overline{v}_{mr} \\ \delta\overline{\omega}_{m} \end{bmatrix} = \begin{bmatrix} 0_{n_\ell} & 0_{n_\ell} & 0_{n_m} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_\ell} & K\Pi_m(\tilde{q}_{mv}, t) & 0_{n_m} \\ 0_{n_\ell} & -\Pi_m(\tilde{q}_{mv}, t)K^\top & 0_{n_m} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_\ell} & 0_{n_m} & 0_{n_m} \end{bmatrix} \begin{bmatrix} \delta\tilde{y}_{\ell v} \\ \delta\tilde{y}_{mv} \end{bmatrix} + \begin{bmatrix} 0_{n_\ell} & 0_{n_m} \\ I_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & 0_{n_m} \\ 0_{n_\ell} & I_{n_m} \end{bmatrix} \delta\omega \tag{5.45}$$

results in the differentially passive system (4.17) *with storage function*

$$W(\tilde{x}_v, \delta\tilde{x}_v, t) = W_\ell(\tilde{x}_{\ell v}, \delta\tilde{x}_{\ell v}, t) + W_m(\tilde{x}_{mv}, \delta\tilde{x}_{mv}, t).$$

Proof: It follows directly from the substitution of the interconnection law (5.45) in the variational dynamics (5.39) and (5.42). This is also shown in Proposition 5.2. ■

The statement in Corollary (5.6) suggests that the control design can be performed separately for the links and motors subsystems. Indeed, under the potential energy matching condition (5.8), the virtual FJR (5.7) can be seen as the pH-like links dynamics with input $u_{\ell v}$, interconnected to the pH-like motors dynamics with input u_{mv} through a spring with deflection $\tilde{q}_{mv} = q_m - q_{md}$ and stiffness constant K. Due to the links and motors dynamics are fully-actuated by their corresponding inputs, the control scheme for rigid robots of Proposition 4.2 can be applied to both subsystems. However, the differential passivity preserving interconnection law has to be constructed carefully because under condition (5.8) the links dynamics must be interconnected to the spring (system) through the term $K\tilde{q}_{md}$ (acting in the dynamics of $\sigma_{\ell v}$). For a *spring-system* (van der Schaft and Jeltsema 2014) the effort (output/force) is modulated by the flow (input/velocity), then the interconnection term $K\tilde{q}_{md}$ is generated via the extra term $\overline{v}_{mr}$ in (5.10). It follows that the tracking problem of each subsystem and the differential passivity preserving interconnection law are can be constructed by the control variables simultaneously.

Remark 5.7. *The statement in Corollary 5.6 is closely related to the main result in the work of (Jardón-Kojakhmetov et al. 2016), where a tracking controller for FJRs was developed using the singular perturbation approach. Under time-scale separation assumptions, it is shown that controller design can be performed in a* composite manner *as* $u = u_s + u_f$*, where the links dynamics* slow *controller* u_s *and the motors dynamics* fast *controller* u_f *can be designed separately. Both systems, the slow and fast, are fully-actuated and standard controllers for rigid robots can be applied as long as exponential stability can be guaranteed. For instance the controller in* (4.5).

In this work *none explicit assumption of time scale separation* was made in the design process. Nevertheless, due to condition (5.8), it is required that the motors position error dynamics converges "faster" than the links one since $K\zeta_v = u_{\ell v} + K\tilde{q}_{mv}$. In this sense, the singular perturbation approach can be used in order to develop a gain tuning scheme for adjusting the convergence rate of the closed-loop system (5.30).

5.5.3 Passivity properties

It is easy to verify that the map $\omega \mapsto \tilde{y}_v$ is cyclo-passive with storage function (5.37) for the closed-loop system (5.36); in fact strictly passive due to conditions (4.4) and (5.11). This is a direct consequence of the pH-like structure preserving conditions (5.35) which represents the generalization to FJRs of Corollary 4.5. Furthermore, passivity of the system (5.36) is independent of the properties on $\phi_\ell(\tilde{q}_{\ell v})$ and Λ_m.
Notice that $\Pi_\ell\phi_\ell(\tilde{q}_{\ell v})$ should be designed carefully since passivity of system (5.36) does not necessarily imply differential passivity; the converse is true.

In the following result necessary and sufficient conditions on $\phi_\ell(\tilde{q}_{\ell v})$ and $\phi_m(\tilde{q}_{mv}) = \Lambda_m\tilde{q}_{mv}$ are given in order to guarantee differential passivity and strict passivity of the closed-loop system (5.36) simultaneously. To this end, recall Definition 4.8, then Corollary 4.9 is adapted for FJRs as

Lemma 5.8. *If* $\Pi_\ell(\tilde{q}_{\ell v}, t)$ *and* $\Pi_m(\tilde{q}_{mv}, t)$ *are constant in* (4.4) *and* (5.11)*, respectively. Then, the maps* $\chi_\ell(\tilde{q}_{\ell v}) = \Pi_\ell\phi_\ell(\tilde{q}_{\ell v})$ *and* $\chi_m(\tilde{q}_{mv}) = \Pi_m\tilde{q}_{mv}$ *are strictly incrementally passive.*

Proof: Same procedure as in the proof of Corollary 4.9 for each ϕ_ℓ and ϕ_m. ■

Similar to the fully-actuated case, there may exist incrementally passive maps $\chi_\ell(\tilde{q}_{\ell v})$ and $\chi_m(\tilde{q}_{mv})$ which do not satisfy inequalities (4.4) and (5.11), respectively. The following result extents the necessary and sufficient conditions in Proposition 4.10 to the FJRs in order to guarantee both properties.

Proposition 5.9. *Consider the maps* $\chi_\ell(\tilde{q}_{\ell v}) = \Pi_\ell\phi_\ell(\tilde{q}_{\ell v})$ *and* $\chi_m(\tilde{q}_{mv}) = \Pi_m\tilde{q}_{mv}$*, with* Π_ℓ *and* Π_m *symmetric positive definite and constant. Inequalities* (4.4) *and* (5.11) *are* *satisfied if and only if the following condition holds:*

$$(\tilde{q}_{kv,2} - \tilde{q}_{kv,1})^\top [\chi_k(\tilde{q}_{kv,2}) - \chi_k(\tilde{q}_{kv,1})] \geq 2\beta_{\tilde{q}_{kv}}(\tilde{q}_{kv,2} - \tilde{q}_{kv,1})^\top \Pi_k(\tilde{q}_{kv,2} - \tilde{q}_{kv,1}) > 0, \quad (5.46)$$

for every $\tilde{q}_{kv,1}, \tilde{q}_{kv,2}$ *and all* $k \in \{\ell, m\}$.

Proof: The sufficiency follows from Lemma 5.8. The necessity follows in the same manner as in the proof of Proposition 4.10 with $\chi_k(\tilde{q}_{kv,2})$ for every $\tilde{q}_{kv,1}, \tilde{q}_{kv,2}$ and all $k \in \{\ell, m\}$ ■

By Remark 4.11, if the conditions of Proposition 5.9 are not satisfied, using Lemma 5.8 we still can find (incrementally/shifted) passive maps χ_ℓ and χ_m that make (5.37) a Lyapunov function for system (5.36) with minimum at the origin. However, under Lemma 5.8 it is not possible to ensure that there exist a unique steady-state trajectory of (5.36) since contraction conditions (4.4) and (5.11) are not necessarily satisfied. This in turn implies that it may not converge to the desired trajectory $q_{\ell d}$ despite the FJR (5.4) in closed-loop with (5.26) being stable.

Remark 5.10 (Existence of a sliding manifold for FJRs)**.** *In Corollary 4.14 it was showed that for a fully-actuated mechanical system there exist a sliding manifold on which the desired trajectory is invariant and attractive. However, since the FJR* (5.4) *is an underactuated mechanical system, this may not be the case in general. The main reason is that the spring's potential force can be compensated only through the actuated degrees of freedom and therefore it can not be eliminated from the links dynamics. However, as shown before, under condition* (5.8) *it is possible to find a suitable motors position reference so that the spring's force output is* $u_{\ell v}$ *when* $\tilde{q}_{mv} = 0_{n_m}$*. In fact this is a necessary and sufficient condition for the existence of an invariant manifold. This can be seen as time-scale separation between the links and motors dynamics.*

Remark 5.11 (Passivity-based sliding controllers for pH-FJRs)**.** *Since the structure of the flexible joint robot* (5.4) *in closed-loop with the* (5.26) *is the same as* (5.36)*, applying Lemma 5.8 for* $\tilde{q}_{\ell v} = \tilde{q}_\ell$ *and* $\tilde{q}_{mv} = \tilde{q}_m$ *results in a large class of passivity-based sliding controllers for the flexible joint robot* (5.4)*. This, however, will not be further explored in this work.*

5.6 Experimental case of study: A FJR of 2 dof

In this section the design procedure and experimental evaluation of two schemes which lie in the family of v-CBC controllers designed in Proposition 5.2 are presented. Each of these tracking controllers exhibit different closed-loop properties with respect to Section 5.5. Furthermore, by Corollary 5.4, the closed-loop variational dynamics structure can be used as a *qualitative tool* during the gain tuning process due to matrices in (5.29) allow to have a *clear physical interpretation* of the design parameters of the controller (5.12) in terms of linear mass-spring-dampers systems, modulated[4] by the state x of the original FJR. This interpretation is as follows: the state dependent and time-varying matrix $\Pi_{\tilde{q}_v}(\tilde{q}_v, t)$ can be understood as the variable-gain matrix of a proportional action that shapes the "potential energy" of the closed-loop variational system (5.32). On the other hand, the terms $\phi(\tilde{q}_v)$ and K_d inject "dissipation" to the whole state of (5.32) through the matrix $\Upsilon(\tilde{x}_v, t)$ in (5.29), where the contribution of $\phi(\tilde{q}_v)$ is modulated by $\Pi_{\tilde{q}_v}(\tilde{q}_v, t)$.

Similar to the fully-actuated case in Section 4.3.1, considering the original (error) state $\tilde{x}$, let the above controllers family be denoted as

$$(\Pi(\tilde{q}, t), K_d, \phi(\tilde{q}))\text{-controller.}$$

For the experiments we consider $t \mapsto q_{\ell d}(t) = [\sin(t), \dots, \sin(t)]^\top \in \mathcal{Q}_\ell$ as a desired links trajectory and $\Pi(\tilde{q}, t) = \Lambda := \mathrm{diag}\{\Lambda_\ell, \Lambda_m\}$ as the position contraction metric, where Λ_ℓ and Λ_m are constant[5] and positive definite diagonal matrices. The controllers will be implemented on the Quanser system described in Section 4.3.1 in FJR configuration.

5.6.1 A saturated-type $(\Lambda, K_d, \mathbf{Tanh}(\tilde{q}_v))$-controller

This is an example of Corollary 5.5 where only the pH-like variational structure in (3.42) is preserved in the closed-loop. This is the FJRs version of the scheme in Section 4.3.3.

Controller construction

Since conditions on Π_q and K_d are already given (both constant), the constructive procedure is reduced to finding $\phi_\ell(\tilde{q}_{\ell v})$ and $\phi_m(\tilde{q}_{mv})$ such that inequalities in (4.4) and (5.11) hold, or equivalently to find a function $\phi_1(\tilde{q}_v) = [\phi_\ell^\top(\tilde{q}_{\ell v}), \phi_m^\top(\tilde{q}_{mv})]^\top$ such that

$$-\Lambda \frac{\partial \phi_1}{\partial \tilde{q}_v}(\tilde{q}_v) - \frac{\partial \phi_1^\top}{\partial \tilde{q}_v}(\tilde{q}_v)\Lambda \leq -2\beta_{\tilde{q}}\Lambda. \tag{5.47}$$

[4]These linear mass-spring-dampers systems have state x_v, and are modulated by the "parameter" x in the sense that their corresponding state space is given by $T_x\mathcal{X}$.

[5]Constructing non-constant contraction metrics is not easy in general. However, some procedures have been proposed. The interested reader is referred to the works (Sanfelice and Praly 2015) and (Kawano and Ohtsuka 2017), where the construction of a state-dependent matrix $\Pi_{\tilde{q}_v}(\tilde{q}_v, t)$ is considered.

Corollary 5.12. *Consider the operators introduced in* (4.50) *and* $\phi_1(\tilde{q}_v) := \Lambda Tanh(\tilde{q}_v)$. *Then, the hypotheses in Corollary 5.5 hold and inequality* (5.47) *is satisfied with*

$$\beta_{\tilde{q}} = \frac{\lambda_{\min}(\Lambda^2) \cdot \lambda_{\min}(SECH^2(\tilde{q}_v))}{\lambda_{\max}(\Lambda)}. \tag{5.48}$$

Proof: The Jacobian of $\phi_1(\tilde{q}_v)$ is given by $\frac{\partial \phi}{\partial \tilde{q}_v}(\tilde{q}_v) = \mathrm{SECH}^2(\tilde{q}_v)$. Since both Λ and SECH$(\cdot)$ are diagonal by construction, inequality (5.47) becomes

$$-\Lambda^2\mathrm{SECH}^2(\tilde{q}_v) - \mathrm{SECH}^2(\tilde{q}_v)\Lambda^2 \leq -\beta_{q_v}\Lambda \iff -2\Lambda^2\mathrm{SECH}^2(\tilde{q}_v) \leq -\beta_{q_v}\Lambda. \tag{5.49}$$

By Rayleigh's theorem, (5.49) is upper bounded as follows

$$-2\Lambda^2\mathrm{SECH}^2(\tilde{q}_v) \leq -2\lambda_{\min}(\Lambda^2)\lambda_{\min}(\mathrm{SECH}^2(\tilde{q}_v))I_n \leq -2\frac{\lambda_{\min}(\Lambda^2) \cdot \lambda_{\min}(\mathrm{SECH}^2(\tilde{q}_v))}{\lambda_{\max}(\Lambda)}\Lambda. \tag{5.50}$$

The product ΛSECH$(\cdot)$ commutes and therefore is symmetric. ∎

Notice that as in (4.54) for the rigid robot case, $\phi_1(\tilde{q}_v)$ is a conservative vector field. Indeed, the corresponding potential function is given as

$$P_v(\tilde{q}_v) = \int_0^{\tilde{q}_v} \phi_1(\xi)\mathrm{d}\xi = \sum_{k=1}^{n_\ell} \lambda_k \ln(\cosh(\tilde{q}_{\ell v,k})) + \sum_{k=1}^{n_m} \lambda_k \ln(\cosh(\tilde{q}_{mv,k})). \tag{5.51}$$

This function can also be interpreted as the closed-loop potential energy on the manifold $\sigma_v = 0_n$ due to the structure of the ideal sliding dynamics in (4.42).

Remark 5.13. *Since the range of sech*$(\cdot)$ *is* $(0, 1]$, *the vector field* $\phi_2(\tilde{q}_v) = \Lambda\tilde{q}_v$ *also satisfies the inequalities in* (4.4) *and* (5.11) *with rate*

$$\beta_{\tilde{q}} = \frac{\lambda_{min}(\Lambda^2)}{\lambda_{max}(\Lambda)}. \tag{5.52}$$

Moreover, with $\phi_2(\tilde{q}_v) = \Lambda\tilde{q}_v$ *condition* (5.35) *holds. The corresponding pH-like form* (3.40) is preserved *and the Hamiltonian function in* (5.37) *is*

$$\tilde{H}_v(\tilde{x}_v, x) = \frac{1}{2}\tilde{q}^\top\Lambda\tilde{q} + \frac{1}{2}\sigma^\top M^{-1}(q)\sigma. \tag{5.53}$$

It follows that the scheme with $\phi_2(\tilde{q}_v)$ *is a structure preserving* passivity-based sliding controller *for the original JFR in* (5.4). *This scheme is in fact the generalization of the one developed in Section 4.3.2 for the rigid robot; reported in (Reyes-Báez et al. 2018).*

Experimental results

The experimental results on the RR robot of Figure 4.1 in closed-loop system with this saturated-type $(\Lambda, K_d, \phi_1(\tilde{q}_v))$-controller are shown in Figure 5.2. The gain matrices are $\Lambda_\ell = \text{diag}\{55, 30\}$, $\Lambda_m = \text{diag}\{70, 60\}$, $K_{\ell d} = \text{diag}\{15, 10\}$ and $K_{md} = \text{diag}\{10, 5\}$.
On the upper two figures, the time responses of $\tilde{q}_\ell$ and q_m are shown. On the left upper plot $\tilde{q}_\ell$ and q_m are compared with the desired trajectory $q_{\ell d}$; it can be seen that links and motors positions converge to $q_{\ell d}$, but only practically since there are steady-state errors. These can be better observed in the upper right plot, where the error variables are shown.

On the lower left plot of Figure 5.2, it is observed that the time response of the momentum error variables also converge practically to zero and there is noise in the signals. The main reason of these behaviors is that the velocity (and hence the momentum) are computed numerically through a filter block in Simulink; which causes some noise.

Even though the family of controllers in Proposition 5.2 requires the computation of the second and third derivatives of q_ℓ due to the definition of p_{mr} in (5.10), it was possible to implement the controller without them by employing directly the dynamical equations in (5.4). In fact, the control signals shown in the right-lower plot in Figure 5.2 do not show much noise effects.

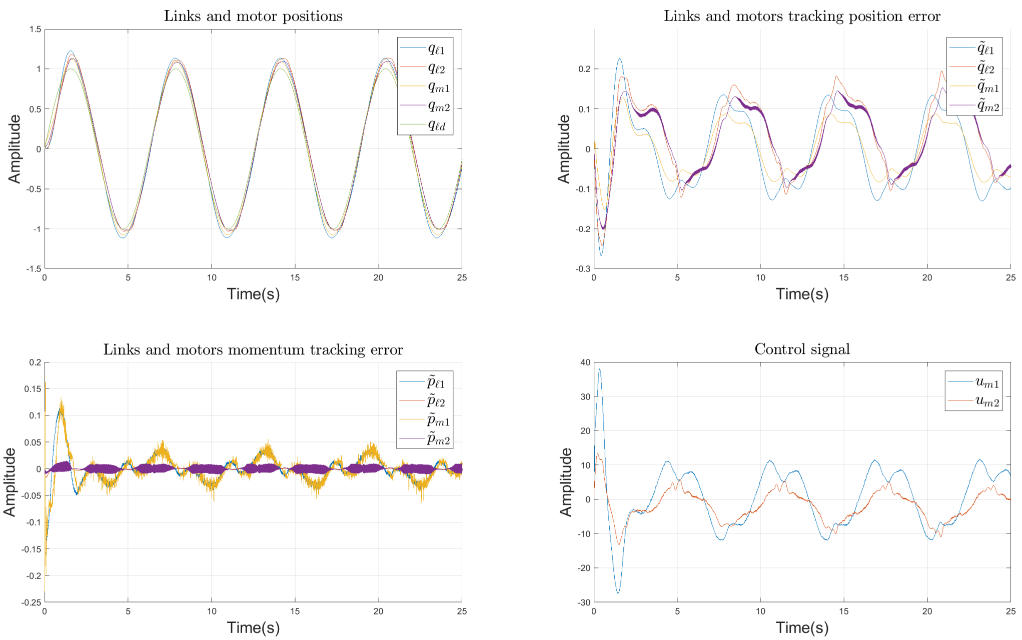

Figure 5.2: Closed-loop and control signals with the $(\Lambda, K_d, \text{Tanh}(\tilde{q}_v))$-controller.

5.6.2 A v-CBC $(\Lambda, K_d, \phi_{\mu_1}(\cdot))$-controller

Now, the scheme in Section 4.3.4 is extended to the flexible-joints case, where the equivalence relation between condition (2.14) in the direct differential Lyapunov method of Theorem 2.6 and its counterpart for *generalized Jacobian* in (2.22) in terms of matrix measures is exploited. In this specific case, the matrix measure associated to the $\|\Theta x\|_1$ norm for a given matrices $\Theta, A \in \mathbb{R}^{p\times p}$ defined in Table 2.1 is considered.

Controller construction

The generalized Jacobian for $\phi_3(\tilde{q}_v) = [\phi_\ell^\top(\tilde{q}_{\ell v}), \phi_m^\top(\tilde{q}_{mv})]^\top$ in this case is

$$\overline{J}(\tilde{q}_v, t) = \Theta\frac{\partial\phi_3}{\partial\tilde{q}_v}(\tilde{q}_v)\Theta^{-1} = \begin{bmatrix} -\frac{\partial\phi_{\ell 1}}{\partial\tilde{q}_{\ell v1}}(\tilde{q}_{\ell v}) & -\frac{\theta_1}{\theta_2}\frac{\partial\phi_{\ell 1}}{\partial\tilde{q}_{\ell v2}}(\tilde{q}_{\ell v}) & 0_{n_\ell} & 0_{n_\ell} \\ -\frac{\theta_2}{\theta_1}\frac{\partial\phi_{\ell 2}}{\partial\tilde{q}_{\ell v1}}(\tilde{q}_{\ell v}) & -\frac{\partial\phi_{\ell 2}}{\partial\tilde{q}_{\ell v2}}(\tilde{q}_{\ell v}) & 0_{n_\ell} & 0_{n_\ell} \\ 0_m & 0_m & -\frac{\partial\phi_{m1}}{\partial\tilde{q}_{mv1}}(\tilde{q}_{mv}) & -\frac{\theta_3}{\theta_4}\frac{\partial\phi_{m1}}{\partial\tilde{q}_{mv2}}(\tilde{q}_{mv}) \\ 0_m & 0_m & -\frac{\theta_4}{\theta_3}\frac{\partial\phi_{m2}}{\partial\tilde{q}_{mv1}}(\tilde{q}_{mv}) & -\frac{\partial\phi_{m2}}{\partial\tilde{q}_{mv2}}(\tilde{q}_{mv}) \end{bmatrix}, \tag{5.54}$$

where $\Lambda = \Theta^\top\Theta$, for matrix $\Theta = \mathrm{diag}\{\theta_1, \theta_2, \theta_3, \theta_4\} > 0_n$. The matrix measure for (5.54) is

$$\begin{aligned}\mu_1\left(\overline{J}(\tilde{q}_v, t)\right) = \max\Bigg\{ &-\frac{\partial\phi_{\ell 1}}{\partial\tilde{q}_{\ell v1}}(\tilde{q}_v) + \left|\frac{\theta_2}{\theta_1}\frac{\partial\phi_{\ell 2}}{\partial\tilde{q}_{\ell v1}}(\tilde{q}_v)\right|, -\frac{\partial\phi_{\ell 2}}{\partial\tilde{q}_{\ell v2}}(\tilde{q}_v) + \left|\frac{\theta_1}{\theta_2}\frac{\partial\phi_{\ell 1}}{\partial\tilde{q}_{\ell v2}}(\tilde{q}_v)\right|, \\ &-\frac{\partial\phi_{m1}}{\partial\tilde{q}_{mv1}}(\tilde{q}_v) + \left|\frac{\theta_4}{\theta_3}\frac{\partial\phi_{m2}}{\partial\tilde{q}_{mlv1}}(\tilde{q}_v)\right|, -\frac{\partial\phi_{m2}}{\partial\tilde{q}_{mv2}}(\tilde{q}_v) + \left|\frac{\theta_3}{\theta_4}\frac{\partial\phi_{m1}}{\partial\tilde{q}_{mv2}}(\tilde{q}_v)\right|\Bigg\}.\end{aligned} \tag{5.55}$$

Thus, the contraction condition (5.47) is equivalent to

$$\mu_1(\overline{J}(\tilde{q}_v, t)) \leq -2\beta_{\tilde{q}_v}, \tag{5.56}$$

where $2\beta_{\tilde{q}_v} := \min\{c_1^2, c_2^2, c_3^2, c_4^2\}$, with c_1, c_2, c_3, c_4 positive constants satisfying

$$\begin{aligned} &\overline{J}_{11}(\tilde{q}_v) + |\overline{J}_{21}(\tilde{q}_v)| < -c_1^2; \quad \overline{J}_{22} + |\overline{J}_{12}| < -c_2^2; \\ &\overline{J}_{33}(\tilde{q}_v) + |\overline{J}_{43}(\tilde{q}_v)| < -c_3^2; \quad \overline{J}_{44} + |\overline{J}_{34}| < -c_4^2. \end{aligned} \tag{5.57}$$

Corollary 5.14. *Let $\phi_3(\tilde{q}_v)$ be defined by*

$$\phi_3(\tilde{q}_v) = \begin{bmatrix} \phi_{\ell 1}(\tilde{q}_{\ell v}) \\ \phi_{\ell 2}(\tilde{q}_{\ell v}) \\ \phi_{m1}(\tilde{q}_{mv}) \\ \phi_{m2}(\tilde{q}_{mv}) \end{bmatrix} = \begin{bmatrix} (1+\kappa_1)\tilde{q}_{\ell v1} + \frac{\theta_2}{\theta_1}tanh(\tilde{q}_{\ell v2}) \\ \frac{\theta_1}{\theta_2}tanh(\tilde{q}_{\ell v1}) + (1+\kappa_2)\tilde{q}_{\ell v2} \\ (1+\kappa_3)\tilde{q}_{mv1} + \frac{\theta_4}{\theta_3}tanh(\tilde{q}_{mv2}) \\ \frac{\theta_3}{\theta_4}tanh(\tilde{q}_{mv1}) + (1+\kappa_4)\tilde{q}_{mv2} \end{bmatrix}, \tag{5.58}$$

where $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ *are strictly positive constants. Then, condition* (5.56) *is satisfied with* $c_1^2 = \kappa_1, c_2^2 = \kappa_2, c_3^2 = \kappa_3$ *and* $c_4^2 = \kappa_4$ *.*

Proof: We shall verify that conditions (5.57) hold for (5.58). Indeed, since the range of $\mathrm{sech}(\cdot)$ is $(0, 1]$, we have that

$$\begin{aligned}
\overline{J}_{11} + |\overline{J}_{21}| \leq -c_1^2 &\iff -(1+\kappa_1) + \left|\mathrm{sech}^2(\tilde{q}_{\ell v1})\right| \leq -\kappa_1 = -c_1^2, \\
\overline{J}_{22} + |\overline{J}_{12}| \leq -c_2^2 &\iff -(1+\kappa_2) + \left|\mathrm{sech}^2(\tilde{q}_{\ell v2})\right| \leq -\kappa_2 = -c_2^2. \\
\overline{J}_{33} + |\overline{J}_{43}| \leq -c_3^2 &\iff -(1+\kappa_3) + \left|\mathrm{sech}^2(\tilde{q}_{mv1})\right| < -\kappa_3 = -c_3^2, \\
\overline{J}_{44} + |\overline{J}_{34}| \leq -c_2^2 &\iff -(1+\kappa_4) + \left|\mathrm{sech}^2(\tilde{q}_{mv2})\right| \leq -\kappa_4 = -c_4^2. \blacksquare
\end{aligned} \tag{5.59}$$

■

With this scheme neither the structure of (3.40) nor the variational one of (3.42) are preserved. Nevertheless, uniform global exponential convergence to $q_{\ell d}$ is still guarantee. Interestingly, in this scheme the convergence rate $\beta_{\tilde{q}_v}$ does not depend on gain Λ, which give extra freedom in the tuning process. In particular, when constrained to the manifold $\sigma_v = 0_n$, the convergence to $q_{\ell d}$ can be accelerated by the gain $\kappa_i, i \in \{1, \ldots, 4\}$. Also, it is worth to mention that in this case, $\phi_3(\tilde{q}_v)$ does not represent a potential force since it is not conservative.

Experimental results

For the experiment with this controller, we consider the following specifications: $\kappa_1 = 10, \kappa_2 = 8$, $\theta_1 = \sqrt{\Lambda_{\ell,11}}$, $\theta_2 = \sqrt{\Lambda_{\ell,22}}$, $\theta_3 = \sqrt{\Lambda_{m,11}}$ and $\theta_4 = \sqrt{\Lambda_{m,22}}$ with the same gain matrices Λ_ℓ, Λ_m, $K_{\ell d}$ and K_{md} of the previous experiment.
The closed-loop time response is shown in Figure 5.3. At first stage we can observe that the performance with respect to the previous controller is improved; this is mainly attributed to the gains $\kappa_i, i \in \{1, \ldots, 4\}$.
Indeed, on the left upper plot we can see how the links and motors positions *almost* superimpose the desired links trajectory $q_{\ell d}$. This can be appreciated better on the upper-right plot where the error variables are shown; we observe that we still have only *practical convergence* since there is steady-state errors, but these are considerably reduced with respect to the precious scheme as well as the overshoot in the transient time interval. We also observe some noise in the motors positions.
On the left lower plot we see the time response of the momentum error variables which have considerably decreased with respect to the previous controller. In fact, as it may be

expected the overshoot during the transient time has decreased as well as the steady state momentum errors which amplitudes, expecting $\tilde{p}_{m1}$, is of the order of 10^{-2}. Here we still have the noise problem due to the numerical computation of the momentum feedback and in this case also the control effort.
On the right lower plot, we see that in this case the overshoot of the control signals has increased but the steady-state signals amplitude is more less the same but with a *rms* value added. This is the expected price to pay after adding an extra control gain.

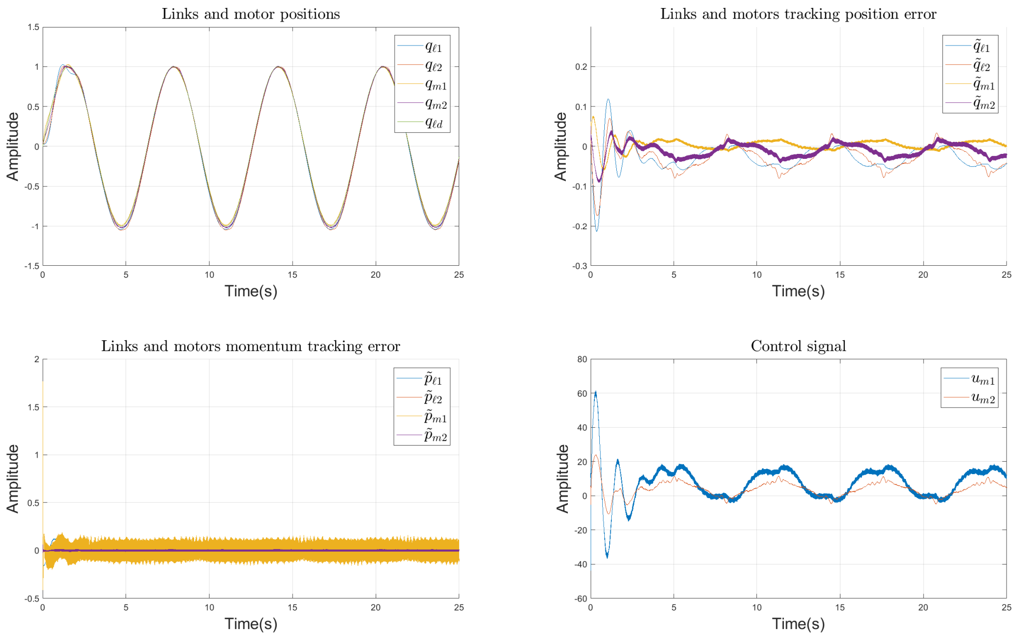

Figure 5.3: Closed-loop and control signals with the $(\Lambda, K_d, \phi_{\mu_1}(\tilde{q}_v))$-controller.

5.7 Conclusions and future research

5.7.1 Conclusions

In this work we have proposed a family of virtual-contraction based controllers that solve the standard trajectory tracking problem of FJRs modeled as port-Hamiltonian systems; that give global exponential convergence. The design procedure is based on the notions of differential passivity and of virtual systems. One of the main advantages with respect to the existing literature is that we do not need to make any assumption on time scale separation since our scheme is not based on singular-perturbation techniques. Instead, by imposing a matching condition between the actual and target *linear* spring's potential energies, we ensure that the torque supplied by the spring to the links dynamics makes their position to track the desired trajectory. However, this limits the class of possible (nonlinear) springs that could be considered and it is the reason why we need the links position second and third derivative available for feedback.

By using the FJR internal workless forces, we have constructed a suitable corresponding virtual system that was used in the v-CBC procedure. Remarkably, this virtual system preserved the passivity property of the original pH-FRJ system.
The developed PD+ type family of v-dPBC tracking has three design "parameters" which can give different structural properties of the closed-loop virtual system. These properties are pH-like structure preserving, variational pH-like structure preserving or none. We also have shown that closed-loop virtual system is nothing but the resulting system of the feedback interconnection of two differentially passive fully-actuated systems. Moreover, we have given if and only if conditions under which the drift vector field of the closed-loop position error dynamics is passive and incrementally passive.

We have constructed two *novel* v-CBC tracking schemes using the Riemannian and matrix measure approaches. With the Riemannian approach we have constructed a saturated-type controller using the hyperbolic tangent function that preserves the variational pH-like structure. On the other hand, using the matrix measure we have constructed a rather exotic controller that does not preserve any structure, but has extra gains that give more freedom for the tuning process. The efficacy of our controllers is verified in the experiments using the planar RR robot by from Quanser.

5.7.2 Future research

There are many future research lines motivated by both, theory and practice. Among these, the followings are highlighted: the first one, to design observers for output feedback in a differential passivity preserving manner. This may solve the problem of availability of higher order derivatives.
Secondly, to explore alternative control design methods that guarantee the differential passivity property of the closed-loop virtual system apart of the control differential Lyapunov functions used here. For instance, the timed-IDA-PBC on the virtual systems.
Finally, to develop an algorithm that gain tunning via singular perturbations techniques. Explore other matrix measures for the development of new tracking controllers.

Chapter 6

Virtual contraction based control of port-Hamiltonian marine craft

"When you can't change the direction of the wind, adjust your sails."

- H. Jackson Brown Jr.

In this chapter a family of v-CBC controllers which solve the tracking problem of marine craft in the port-Hamiltonian framework is considered. Since the marine craft are modeled as a *rigid body* on moving frames, the controller construction is performed in two scenarios. That is, pH models in body-fixed and inertial frames are considered. In particular, it is shown how the *intrinsic structure* of pH models and their *workless* forces can be exploited to construct virtual control systems for marine craft in both frames. Later, by using these systems, the control design results of the previous chapters are extended to the rigid body case. Finally, the closed-loop system's performance is evaluated on numerical simulations.

6.1 Introduction

The dynamic models of marine craft and hydrodynamic forces possess intrinsic passivity properties inherited from their physical nature (Fossen 1994, Fossen 2011). These properties have been widely used for motion control design of ships and underwater vehicles; see for instance the works (Fossen and Berge 1997, Sorensen and Asgeir 1995, Woolsey and Leonard 2002) and references therein.

In the same spirit, port-Hamiltonian models for marine craft have been proposed and used for passivity-control design where the closed-loop dynamics preserve the port-Hamiltonian structure; for further details see (Donaire and Perez 2012, Donaire et al. 2017) and references therein. Specifically, in (Donaire et al. 2017), two marine craft pH models are presented, one in a body-fixed frame and another in an inertial frame[1].

[1] From a practical point of view, the relation between these two models is very useful because the attitude and velocities are measured by IMU (Inertial Measurement Unit) and GPS (Global Position System) sensors.

The body-frame pH model is later used for designing a passivity based tracking control scheme. However, since the inertia matrix of the pH model in the inertial frame is not constant, the trajectory tracking control problem is not solved.

In order to extend the design method used for the body-fixed pH model in (Donaire et al. 2017) to the inertial frame; as intermediate step; it is necessary to construct a canonical transformation such that in the new coordinates the inertia matrix in the inertial frame is constant. This turns messy since it requires the solution of a set of ODEs. See (Fujimoto et al. 2003, Venkatraman et al. 2010) and (Romero, Ortega and Sarras 2015) for further details for the transformation construction.

In (Reyes-Báez, Donaire, van der Schaft, Jayawardhana and Perez 2019), a novel solution to the tracking control problem of marine craft in the pH framework is presented via the v-CBC technique Instead of solving the aforementioned set of ODEs. Specifically, the control scheme in (Reyes-Báez et al. 2017) is extended to the rigid body-case.

6.2 Craft's Newton-Euler and quasi-Lagrange models

The goal of this section is to present the marine craft EL models developed by (Fossen 2011) in terms of its workless forces. This form will be instrumental for the construction of associated virtual systems in both the body and inertial frames.

The standard notation of *SNAME (1950)* for marine vessels is adopted. From a guidance, navigation and control point of view, in the modeling of marine craft, four different coordinate frames are considered: the Earth Centered inertial (ECI) frame $\{i\} = \{x_i, y_i, z_i\}$, whose origin O_i is located at the center of mass of the Earth; the Earth-centered-Earth-fixed (ECEF) frame $\{e\}$ that rotates with the Earth; the North-East-Down (NED) frame $\{n\} = \{x_n, y_n, z_n\}$ with origin O_n defined relative to the Earth's reference ellipsoid (WGS84); and the body frame $\{b\} = \{x_b, y_b, z_b\}$, with origin O_b, which is a moving coordinate frame that is fixed to the craft. See Figure 6.1.

For a marine craft, the origin O_b is usually chosen to coincide with a point midship CO in the water line, see Figure 6.2; while the body-frame axes are chosen to coincide with the principal axes of inertia (Fossen 2011):

- x_b - longitudinal axis (directed from aft to fore)
- y_b - transversal axis (directed to starboard)
- z_b - normal axis (directed from top to bottom)

In this work the following modeling assumptions are taken (Donaire et al. 2017):

Assumption 6.1 (Flat navigation)**.** *The operating radius of a marine craft is limited. We assume $\{n\}$ to be inertial.*

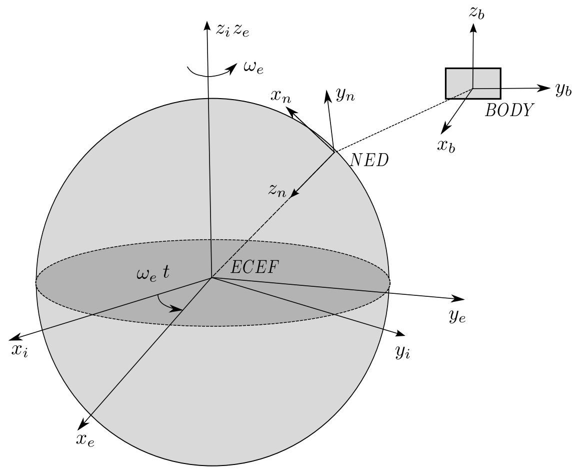

Figure 6.1: ECEF frame rotating with angular rate ω_e with respect to the ECI frame.

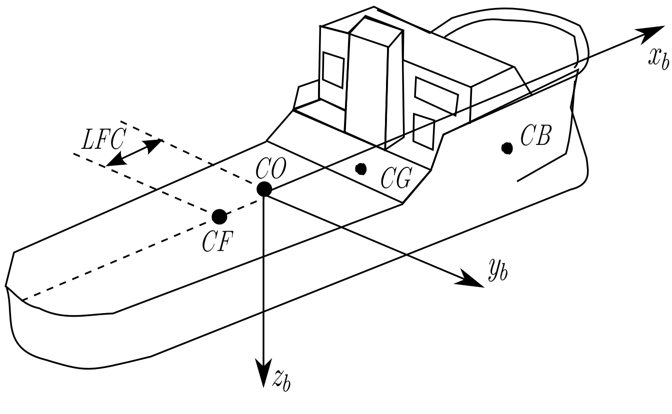

Figure 6.2: Body-fixed frame.

Assumption 6.2 (Maneuvering theory)**.** *The hydrodynamic coefficients are frequency independent (no wave excitation).*

Under the above assumptions, by the Newton-Euler (N-E) approach, the equation of motion in the body-frame are by (Fossen 2011):

$$\begin{aligned} \dot{\eta} &= J(\eta)\nu, \\ M\dot{\nu} + C(\nu)\nu + D(\nu)\nu + g(\eta) &= \tau \end{aligned} \tag{6.1}$$

where $\eta = [\eta_1^\top, \eta_2^\top] \in Q := \mathbb{R}^3 \times S^3$ is the vector of generalized coordinates, where $\eta_1 = [x, y, z]^\top \in \mathbb{R}^3$ and $\eta_2 = [\phi, \theta, \psi]^\top \in S^3$ describe the marine craft's position and orientation (roll, pith, yaw), respectively. The vector $\nu = [\nu_1^\top, \nu_2^\top]^\top$ is the (quasi-)velocity in $\{b\}$, with $\nu_1 = [u, v, w]^\top$(surge, sway, heave) and $\nu_2 = [p, q, r]^\top$ (roll, pith, yaw) rates; and τ are the force and torque inputs.

The matrix function[2] $J(\eta)$ in terms of the Euler angles is given as

$$J(\eta) = \begin{bmatrix} \mathrm{c}\psi \cdot \mathrm{c}\theta & -\mathrm{s}\psi \cdot \mathrm{c}\phi + \mathrm{c}\psi \cdot \mathrm{s}\theta \cdot \mathrm{s}\phi & \mathrm{s}\psi \cdot \mathrm{s}\phi + \mathrm{c}\psi \cdot \mathrm{c}\phi \cdot \mathrm{s}\theta & 0 & 0 & 0 \\ \mathrm{s}\psi \cdot \mathrm{c}\theta & \mathrm{c}\psi \cdot \mathrm{c}\phi + \mathrm{s}\phi \cdot \mathrm{s}\theta \cdot \mathrm{s}\psi & -\mathrm{c}\psi \cdot \mathrm{s}\phi + \mathrm{s}\theta \cdot \mathrm{s}\psi \cdot \mathrm{c}\phi & 0 & 0 & 0 \\ -\mathrm{s}\theta & \mathrm{c}\theta \cdot \mathrm{s}\phi & \mathrm{c}\theta \cdot \mathrm{c}\phi & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \mathrm{s}\phi \cdot \mathrm{t}\theta & \mathrm{c}\phi \cdot \mathrm{t}\theta \\ 0 & 0 & 0 & 0 & \mathrm{c}\phi & -\mathrm{s}\phi \\ 0 & 0 & 0 & 0 & \frac{\mathrm{s}\phi}{\mathrm{c}\theta} & \frac{\mathrm{c}\phi}{\mathrm{c}\theta} \end{bmatrix} \tag{6.2}$$

where $\mathrm{s}(), \mathrm{c}(), \mathrm{t}()$ represent the trigonometric functions $\sin(), \cos(), \tan()$, respectively. The transformation matrix (6.2) is well defined if $\theta \neq \pm\pi/2$, see (Fossen 2011) for details.

The inertia matrix $M = M^\top > 0$ is

$$M := \begin{bmatrix} mI & -m\mathcal{S}_\times(r_g^b) \\ m\mathcal{S}_\times(r_g^b) & I_b \end{bmatrix} \tag{6.3}$$

where m is the total mass due to the craft's mass and fluid added mass, I_b the moment of inertia in $\{b\}$, r_g^b the constant vector between O_b and the center of gravity (CG) in $\{b\}$-coordinates, and the matrix $\mathcal{S}_\times(\cdot)$ is defined as

$$\mathcal{S}_\times(a) = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix}. \tag{6.4}$$

According to Kirchhoff's equations of motion, the Coriolis-centripetal matrix $C(\nu)$ is any matrix satisfying the relation

$$C(\nu)\nu = \begin{bmatrix} \mathcal{S}_\times(\nu_2)\frac{\partial T^b}{\partial \nu_1}(\nu) \\ \mathcal{S}_\times(\nu_2)\frac{\partial T^b}{\partial \nu_2}(\nu) + \mathcal{S}_\times(\nu_1)\frac{\partial T^b}{\partial \nu_1}(\nu) \end{bmatrix}, \tag{6.5}$$

where $T^b(\nu) = \frac{1}{2}\nu^\top M \nu$ is the kinetic (co-)energy; $D(\nu) = D^\top(\nu) > 0$ is the total hydrodynamic damping matrix; and $g(\eta)$ is the vector of hydrostatic forces and torques due to gravity and buoyancy. Clearly the force $F_{gr}^b(\nu) := C(\nu)\nu$ in (6.5) is *workless*; that is

[2]It is possible to use alternative representations for the attitude as quaternions or the special ortogonal Lie group. These are left as open problems for future research.

$\nu^\top F^b_{gr}(\nu) = 0$ for every ν. Indeed, the associated work results in

$$\nu^\top F^b_{gr}(\nu) = \nu_1^\top S_\times(\nu_2)\frac{\partial T}{\partial \nu_1}(\nu) - \nu_2^\top S_\times\left(\frac{\partial T}{\partial \nu_2}(\nu)\right)\nu_2 - \nu_1^\top S_\times(\nu_2)\frac{\partial T}{\partial \nu_1}(\nu) = 0.$$

Following (Chang 2002), any workless force $F(\nu)$ can be expressed as $F(\nu) = S_L(\nu)\nu$ for some skew-symmetric matrix $S_L(\nu)$; see Appendix B.1.3 for further details. It follows that the force is written as $F^b_{gr}(\nu) = S^b_L(\nu)\nu$ with $S^b_L(\nu) = C(\nu)$ skew-symmetric. This was first shown in (Fossen 2011) for system (6.1) with different skew-symmetric $C(\nu)$'s.

6.2.1 A remark on marine craft dynamics in the inertial frame

The body-frame equations of motion (6.1) can be also expressed in the inertial frame $\{n\}$ by performing the kinematic transformation $\dot\eta = J(\eta)\nu$ as follows (Fossen 1994, p.48):

$$M_\eta(\eta)\ddot\eta + C_\eta(\eta,\dot\eta)\dot\eta + D_\eta(\eta,\dot\eta)\dot\eta + g_\eta(\eta) = \tau_\eta \tag{6.6}$$

where

$$\begin{aligned} M_\eta(\eta) &= J^{-\top}(\eta)MJ^{-1}(\eta), \\ C_\eta(\eta,\dot\eta) &= J^{-\top}(\eta)\left[C(J^{-1}\dot\eta) - MJ^{-1}(\eta)\dot J(\eta)\right]J^{-1}(\eta), \\ D_\eta(\eta,\dot\eta) &= J^{-\top}(\eta)D(J^{-1}\dot\eta)J^{-1}(\eta), \\ g_\eta(\eta) &= J^{-\top}(\eta)g(\eta), \\ \tau_\eta &= J^{-\top}(\eta)\tau. \end{aligned} \tag{6.7}$$

Alternatively, the Lagrange equations can be used to derive (6.6) since the vector η qualifies as generalized coordinate in $\{n\}$. Thus, the kinetic (co-)energy in $\{n\}$ is given by

$$T_\eta(\eta,\dot\eta) = \frac{1}{2}\dot\eta^\top M_\eta(\eta)\dot\eta, \tag{6.8}$$

and the potential energy $P(\eta)$ is defined as the solution to

$$J^\top(\eta)\frac{\partial P}{\partial \eta}(\eta) = g(\eta). \tag{6.9}$$

It follows that the vehicle-ambient Lagrangian function is given by(Fossen 1994):

$$L(\eta,\dot\eta) = T_\eta(\eta,\dot\eta) - P(\eta). \tag{6.10}$$

Remark 6.3. *A necessary and sufficient conditions for the existence of $P(\eta)$ satisfying* (6.9) *is guaranteed by Poncare's Lemma (Donaire et al. 2017). This condition is the following*

$$\frac{\partial}{\partial \eta}\left[J^{-\top}(\eta)g(\eta)\right] = \left(\frac{\partial}{\partial \eta}\left[J^{-\top}(\eta)g(\eta)\right]\right)^{\top}.$$

For the dissipative hydrodynamic forces, consider the following Rayleigh function

$$F_R(\eta, \dot{\eta}) = \frac{1}{2}\dot{\eta}^{\top} D_{\eta}(\eta, \dot{\eta})\dot{\eta}. \tag{6.11}$$

In the Euler-Lagrange framework, the Coriolis-centripetal matrix is defined as any matrix $C_{\eta}(\eta, \dot{\eta})$ satisfying the identity (Ortega et al. 2013):

$$C_{\eta}(\eta, \dot{\eta})\dot{\eta} = \dot{M}_{\eta}(\eta)\dot{\eta} - \frac{\partial T}{\partial \eta}(\eta, \dot{\eta}). \tag{6.12}$$

Notice that the following work relation holds (Arimoto and Miyazaki 1984)

$$\dot{\eta}^{\top} \underbrace{\left[\frac{1}{2}\dot{M}_{\eta}(\eta)\dot{\eta} - \frac{\partial T}{\partial \eta}(\eta, \dot{\eta})\right]}_{F_{gr}(\eta, \dot{\eta})} = 0. \tag{6.13}$$

This means that the force $F_{gr}(\eta, \dot{\eta})$ is workless in the sense of (3.6). Substitution of (6.13) in the power associated to (6.12) yields the well-know identity

$$\dot{\eta}^{\top} \underbrace{\left[\dot{M}_{\eta}(\eta) - 2C_{\eta}(\eta, \dot{\eta})\right]}_{N_{\eta}(\eta, \dot{\eta})} \dot{\eta} = 0. \tag{6.14}$$

In fact, identity (6.14) is a particular case of the more general relation

$$r^{\top}\left[\dot{M}_{\eta}(\eta) - 2C_{\eta}(\eta, \dot{\eta})\right] r = 0, \tag{6.15}$$

which holds for any $r \in T_{\eta}Q$. This is a consequence of the relation between the Levi-Civita connection and the Riemannian metric associated to the inertia tensor $M_{\eta}(\eta)$. See Appendix B.1.2 for a detailed treatment on this.

Since $F_{gr}(\eta, \dot{\eta})$ is workless, according to (Chang 2002) there exists a skew-symmetric matrix $S^L_{\eta}(\eta, \dot{\eta})$ such that $F_{gr}(\eta, \dot{\eta}) = S^L_{\eta}(\eta, \dot{\eta})\dot{\eta}$; see Appendix B.1.3. This matrix can be expressed in terms of the Christoffel symbols corresponding to the Levi-Civita connection of $M_\eta(\eta)$ as follows:

$$S^L_{\eta kj}(\eta,\dot\eta) = \frac{1}{2}\sum_{i=1}^{n}\left\{\frac{\partial M_{\eta ki}}{\partial \eta_j}(\eta) - \frac{\partial M_{\eta ij}}{\partial \eta_k}(\eta)\right\}\dot\eta_i. \tag{6.16}$$

or equivalently in explicit terms of $J(\eta)$ as

$$S^L_\eta(\eta,\dot\eta) = J^{-\top}(\eta)CJ^{-1}(\eta) + \frac{1}{2}\left[J^{-\top}(\eta)M\dot J^{-1}(\eta) - (J^{-\top}(\eta)M\dot J^{-1})^\top(\eta)\right]. \tag{6.17}$$

Hence, expression (6.12) takes the form

$$C_\eta(\eta,\dot\eta)\dot\eta = \left[\frac{1}{2}\dot M_\eta(\eta) + S^L_\eta(\eta,\dot\eta)\right]\dot\eta, \tag{6.18}$$

Remark 6.4. *With $S^L_\eta(\eta,\dot\eta)$ as in* (6.17)*, the correspondence between the Coriolis matrices in the Newton-Euler formalism in* (6.7) *and in the Lagrange framework in* (6.12) *is* clear *by relation* (6.18). This is not the case *in (Fossen 1994, p. 54).*

6.3 Marine Craft's port-Hamiltonian modeling

In this part the body-fixed and inertial frames pH models in (Donaire et al. 2017) are recalled. These models are then rewritten in the alternative form (3.26).

6.3.1 Craft's pH model in body-frame and workless forces

The next assumption is made on (6.1) (Donaire and Perez 2012, Donaire et al. 2017):

Assumption 6.5. *There exists a potential function $P : \mathcal{Q} \to \mathbb{R}$ satisfying condition* (6.9).

Under Assumption 6.5, following (Donaire et al. 2017), the marine craft dynamics (6.1) can be expressed in port-Hamiltonian framework as

$$\begin{bmatrix}\dot\eta \\ \dot p^b\end{bmatrix} = \begin{bmatrix}0 & J(\eta) \\ -J^\top(\eta) & -F_2(p^b)\end{bmatrix}\begin{bmatrix}\frac{\partial H}{\partial \eta}(\eta,p^b) \\ \frac{\partial H}{\partial p^b}(\eta,p^b)\end{bmatrix} + \begin{bmatrix}0 \\ I_n\end{bmatrix}\tau, \tag{6.19}$$

where the Hamiltonian function is

$$H(\eta,\dot p^b) = \frac{1}{2}p^{b\top}M^{-1}p^b + P(\eta), \tag{6.20}$$

the *quasi momentum*[3] is defined as

$$p^b = M\nu, \tag{6.21}$$

and $F_2(p^b) = C(M^{-1}p^b) + D(M^{-1}p^b)$. System (6.19) is passive with (6.20) as storage function. We define $E^b(\eta, p^b) := C(M^{-1}p^b)$ as the workless forces matrix for future purposes. Notice for the constant intertia matris M and Hamiltonian function (6.20), the structure of (3.26) coincides with (6.19), where $E^b(\eta, p^b) = S^{Hb}(p^b)$ and $S^{Hb}(p^b) = C(M^{-1}p^b)$.

6.3.2 Craft's pH model in inertial-frame and workless forces

A pH model in inertial coordinates was developed in (Donaire et al. 2017), also under Assumption 6.5. To this end, let p^b be transformed into the *true-momentum* of the generalized coordinates vector η in the frame $\{n\}$ as

$$p = J^{-\top}(\eta)p^b \iff p = M_\eta(\eta, \dot\eta)\dot\eta. \tag{6.22}$$

This defines the diffeomorphism $\psi : T^*Q \to T^*Q$ given by

$$\begin{bmatrix}\eta\\ p\end{bmatrix} = \psi(\eta, p^b) = \begin{bmatrix}\eta\\ J^{-\top}(\eta)p^b\end{bmatrix} \iff x^b = \begin{bmatrix}\eta\\ J^{\top}(\eta)p\end{bmatrix}. \tag{6.23}$$

Then, the dynamics (6.21) in coordinates (η, p) is given by

$$\begin{bmatrix}\dot\eta\\ \dot p\end{bmatrix} = \begin{bmatrix}0 & I\\ -I & -L(\eta, p)\end{bmatrix}\begin{bmatrix}\frac{\partial H_\eta}{\partial \eta}(\eta, p)\\ \frac{\partial H_\eta}{\partial p}(\eta, p)\end{bmatrix} + \begin{bmatrix}0\\ I\end{bmatrix}\tau_\eta, \tag{6.24}$$

where the Hamiltonian function is

$$H_\eta(\eta, p) = \frac{1}{2}p^\top M_\eta^{-1}(\eta)p + P(\eta), \tag{6.25}$$

and the matrix

$$L(\eta, p) = \left[\sum_{i=1}^{n}\frac{\partial J^{-\top}}{\partial \eta_i}J^\top p e_i^\top\right]^\top - \left[\sum_{i=1}^{n}\frac{\partial J^{-\top}}{\partial \eta_i}J^\top p e_i^\top\right] + J^{-\top}F_2(M^{-1}J^\top p)J^{-1}, \tag{6.26}$$

with e_i is the i-th element of the Euclidean canonical base. Indeed, (6.24) is a pH system according to Definition 3.5; therefore energy conservation (passivity) is satisfied.

[3]Since ν are velocities measured in the body-fixed frame $\{b\}$ (*quasi-velocities*), p^b is not the true momentum of η. For details see (Greenwood 2003, p. 193) and (Fossen 2011).

Remark 6.6. *The pH model* (6.24) *is not a simple mechanical pH system since its interconnection matrix can not be written as $J_H(x)$ in* (3.24). *In fact, the interconnection matrix of* (6.24) *defines only an* almost-Poisson structure*, i.e., the associated Poisson bracket does not satisfy the Jacobi identity (see Appendix B.2.1 for details).*

In the following proposition an alternative form of the inertial model (6.24) is presented. In this alternative form, *the workless forces are decoupled* from the gradient of the Hamiltonian function. This will be a *key property* in the definition of the associated virtual control system of (6.24) for the v-CBC design later.

Proposition 6.7. *The pH system* (6.24) *takes the form*

$$\begin{bmatrix}\dot{\eta}\\ \dot{p}\end{bmatrix} = \begin{bmatrix}0 & I\\ -I & -(E_\eta(\eta,p)+D^H_\eta(\eta,p))\end{bmatrix}\begin{bmatrix}\frac{\partial P}{\partial \eta}(\eta)\\ M_\eta^{-1}p\end{bmatrix} + \begin{bmatrix}0\\ J^{-\top}\end{bmatrix}\tau, \tag{6.27}$$

where matrices $E_\eta(q,p)$ and $D^H_\eta(q,p)$ are respectively given by

$$E_\eta(q,p) = S^H_\eta(\eta,p) - \frac{1}{2}\dot{M}_\eta(\eta), \qquad D^H_\eta(q,p) = D_\eta(\eta, M_\eta^{-1}(\eta)p), \tag{6.28}$$

with $S^H_\eta(\eta,p) := S^L_\eta(\eta, M_\eta^{-1}(\eta)p)$ from (6.17).

Proof: The time derivative of (6.22) gives

$$\begin{bmatrix}\dot{\eta}\\ \dot{p}\end{bmatrix} = \left(\begin{bmatrix}I & 0\\ \frac{\partial}{\partial \eta}\left(J^{-\top}(\eta)p^b\right) & J^{-\top}(\eta)\end{bmatrix}\begin{bmatrix}\dot{\eta}\\ \dot{p}^b\end{bmatrix}\right)\Bigg|_{x^b=[\eta^\top,(J^\top(\eta)p)^\top]^\top}. \tag{6.29}$$

The dynamics of p is then given by

$$\dot{p} = -\frac{\partial P}{\partial \eta}(\eta) + \left[\dot{J}^{-\top}(\eta)MJ^{-1} - J^{-\top}(\eta)F_2(J^\top(\eta)p)J^{-1}(\eta)\right]M_\eta^{-1}(\eta)p + \tau_\eta. \tag{6.30}$$

The term $\dot{J}^{-\top}(\eta)MJ^{-1}(\eta)$ in (6.30) can be rewritten as follows

$$\begin{aligned}\dot{J}^{-\top}(\eta)MJ^{-1}(\eta) &= \dot{J}^{-\top}(\eta)MJ^{-1}(\eta) - \frac{1}{2}\dot{M}_\eta(\eta)\dot{\eta} + \frac{1}{2}\dot{M}_\eta(\eta)\dot{\eta},\\ &= -\frac{1}{2}\left[J^{-\top}(\eta)M\dot{J}^{-1}(\eta) - (J^{-\top}(\eta)M\dot{J}^{-1})^\top(\eta)\right] + \frac{1}{2}\dot{M}_\eta(\eta)\dot{\eta}\end{aligned} \tag{6.31}$$

where the force $\dot{M}_\eta(\eta)\dot{\eta}$ computed as follows

$$
\begin{aligned}
\dot{M}_\eta(\eta)\dot{\eta} &= \sum_{i=1}^{n} e_i^\top \dot{\eta} \frac{\partial}{\partial \eta_i}\left[J^{-\top}(\eta)MJ^{-1}(\eta)\right]\dot{\eta},\\
&= \sum_{i=1}^{n} e_i^\top \dot{\eta}\left[\frac{\partial J^{-\top}}{\partial \eta_i}(\eta)MJ^{-1}(\eta) + J^{-\top}(\eta)M\frac{\partial J^{-1}}{\partial \eta_i}(\eta)\right]\dot{\eta},\\
&= \sum_{i=1}^{n}\left[e_i^\top \dot{\eta}\frac{\partial J^{-\top}}{\partial \eta_i}(\eta)J^\top(\eta)J^{-\top}(\eta)MJ^{-1}(\eta)\dot{\eta} + e_i^\top \dot{\eta}J^{-\top}(\eta)MJ^{-1}(\eta)J(\eta)\frac{\partial J^{-1}}{\partial \eta_i}(\eta)\dot{\eta}\right],\\
&= \sum_{i=1}^{n}\left[e_i^\top \dot{\eta}\frac{\partial J^{-\top}}{\partial \eta_i}(\eta)J^\top(\eta)p + e_i^\top pJ(\eta)\frac{\partial J^{-1}}{\partial \eta_i}(\eta)\dot{\eta}\right],\\
&= \sum_{i=1}^{n}\left[\frac{\partial J^{-\top}}{\partial \eta_i}(\eta)J^\top(\eta)pe_i^\top + e_i^\top pJ(\eta)\frac{\partial J^{-1}}{\partial \eta_i}(\eta)\right]\dot{\eta},\\
&= \sum_{i=1}^{n}\left[\frac{\partial J^{-\top}}{\partial \eta_i}(\eta)J^\top(\eta)pe_i^\top + \left(\frac{\partial J^{-\top}}{\partial \eta_i}(\eta)J^\top(\eta)pe_i^\top\right)^\top\right]\dot{\eta},\\
&= \left[\frac{\partial J^{-\top}}{\partial \eta}(\eta)J^\top(\eta)p + \left(\frac{\partial J^{-\top}}{\partial \eta}(\eta)J^\top(\eta)p\right)^\top\right]\dot{\eta}.
\end{aligned}
\tag{6.32}
$$

Substitution of (6.31) in (6.30) yields

$$
\dot{p} = -\frac{\partial P}{\partial \eta} - \left[E_\eta(\eta, p) + D_\eta^H(\eta, p)\right]M_\eta^{-1}(\eta)p + \tau_\eta, \tag{6.33}
$$

which completes the proof. ■

The alternative form (6.27) preserves passivity with (6.25) as storage function.

6.4 Control design procedure via v-CBC

The tracking problem for a marine craft will be solved by means of the v-CBC method described in Subsection 2.2.3. A controller will be designed for both the body-fixed pH system (6.19) and the inertial pH system (6.24). Since the marine craft considered in this work is fully-actuated, the design process and closed-loop properties are very close related to the scheme presented in Chapter 4.

6.4.1 Control design in the body-fixed frame

Step 1: Virtual control system design

The structure of (6.19) motivates the definition of the following associated virtual system in the state (η_v, p_v^b) and parametrized by the trajectory (η, p^b), as

$$\begin{bmatrix} \dot{\eta}_v \\ \dot{p}_v^b \end{bmatrix} = \begin{bmatrix} 0 & J(\eta) \\ -J^\top(\eta) & -(E^b(\eta, p^b) + D(M^{-1}p^b)) \end{bmatrix} \begin{bmatrix} \frac{\partial H_v}{\partial \eta_v}(\eta_v, p_v^b) \\ \frac{\partial H_v}{\partial p_v^b}(\eta_v, p_v^b) \end{bmatrix} + \begin{bmatrix} 0 \\ I_n \end{bmatrix} \tau_v, \tag{6.34}$$

with

$$H_v(\eta_v, p_v^b) = \frac{1}{2} p_v^{b\top} M^{-1} p_v^b + P_v(\eta_v) \tag{6.35}$$

where $P_v(\eta_v)$ also fulfills Assumption 6.5 and $P_v(\eta) = P(\eta)$. Since the inertia matrix M is constant, the virtual system (6.35) is also a pH system and passive with (6.35) as storage function and supply rate $y_{b\eta}^\top \tau_\eta$, uniformly in $(\eta(t), p^b(t))$, for all $t > t_0$; where $y_\eta^b = M^{-1}p_v^b$.

Step 2: Contraction-based control design of the virtual pH system

In the following result the control input τ_v will be designed via the contraction-based backstepping such that (6.34) is differentially passive in the closed-loop. This is close related to Proposition 4.2 since the marine craft considered here is fully-actuated.

Proposition 6.8. *Consider system the virtual system* (6.34) *and a smooth desired trajectory $x_d^b = (\eta_d, p_d^b)$ in $\{b\}$ with $p_d^b = MJ^{-1}(\eta)\dot{\eta}_d$. Let*

$$\tilde{x}_v^b := \begin{bmatrix} \tilde{\eta}_v^b \\ \sigma_v^b \end{bmatrix} = \begin{bmatrix} \eta_v - \eta_d \\ p_v^b - p_r^b \end{bmatrix}, \tag{6.36}$$

be the error coordinates, where the auxiliary momentum reference p_r^b is given by

$$p_r^b(\tilde{\eta}_v^b, t) := MJ^{-1}(\eta)(\dot{\eta}_d - \phi^b(\tilde{\eta}_v^b)), \tag{6.37}$$

$\phi^b : Q \to TQ$ is such that $\phi^b(0) = 0$; and $\Pi_{\eta_v}^b : Q \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ is a positive definite Riemannian metric tensor satisfying the inequality

$$\dot{\Pi}_{\eta_v}^b(\tilde{\eta}_v^b, t) - \Pi_{\eta_v}^b(\tilde{\eta}_v^b, t)\frac{\partial \phi^b}{\partial \tilde{\eta}_v^b}(\tilde{\eta}_v^b) - \frac{\partial \phi^{b\top}}{\partial \tilde{\eta}_v^b}(\tilde{\eta}_v^b)\Pi_{\eta_v}^b(\tilde{\eta}_v^b, t) \leq -2\beta_{\eta_v}^b(\tilde{\eta}_v^b, t)\Pi_{\eta_v}^b(\tilde{\eta}_v^b, t), \tag{6.38}$$

with $\beta_{\eta_v}^b(\tilde{\eta}_v^b, t) > 0$, uniformly. Assume also that the i-th row of $\Pi_{\eta_v}^b(\tilde{\eta}_v^b, t)$ is a conservative *vector field*[4]*. Consider also the composite control law given by*

$$\tau_v(x_v^b, x^b, t) := \tau_v^{ff}(x_v^b, x^b, t) + \tau_v^{fb}(x_v^b, x^b, t) + \omega^b, \tag{6.39}$$

where $x^b = (\eta, p^b)$, $x_v^b = (\eta_v, p_v^b)$ *and*

$$\begin{aligned} \tau_v^{ff} &= \dot{p}_r^b + J^\top(\eta)\frac{\partial P_v}{\partial \eta_v} + [E^b(\eta, p^b) + D(M^{-1}p^b)]M^{-1}p_r^b, \\ \tau_v^{fb} &= -\int_0^{\tilde{\eta}_v^b} \Pi_{\eta_v}^b(\xi, t)d\xi - K_d M^{-1}\sigma_v^b, \end{aligned} \tag{6.40}$$

with gain matrix $K_d > 0$ *and* ω^b *is an external input. Then,*

1. *system* (6.34) *in closed-loop with the controller* (6.39) *is differentially passive from* $\delta\omega^b$ *to* $\delta\overline{y}_{\sigma_v}^b = J(\eta)M^{-1}\delta\sigma_v^b$ *with respect to the differential storage function*

$$W_{x^b}(\tilde{x}_v^b, \delta\tilde{x}_v^b, t) = \frac{1}{2}\delta\tilde{x}_v^{b\top}\begin{bmatrix}\Pi_{\eta_v}^b(\tilde{\eta}_v^b, t) & 0 \\ 0 & M^{-1}\end{bmatrix}\delta\tilde{x}_v^b. \tag{6.41}$$

2. *the closed-loop variational dynamics preserves the structure of* (2.24)*, with*

$$\begin{aligned} \Pi^b(\tilde{x}_v^b, t) &= diag\{\Pi_{\eta_v}^b(\tilde{\eta}_v^b, t), M^{-1}\}, \\ \Upsilon^b(\tilde{x}_v^b, t) &= diag\left\{\frac{\partial \phi^b}{\partial \tilde{\eta}_v^b}(\Pi_{\eta_v}^b)^{-1}(\tilde{\eta}_v^b, t), D(M^{-1}p^b) + K_d\right\}, \\ \Xi^b(\tilde{x}_v^b, t) &= \begin{bmatrix}0 & I \\ -I & -S_H^b(p^b)\end{bmatrix}; \quad \Psi^b(\tilde{x}_v^b, t) = \begin{bmatrix}0 \\ I\end{bmatrix}. \end{aligned} \tag{6.42}$$

Proof: The proof follows in the same manner as in the one of Proposition 4.2. The virtual system (6.34) is explicitly given as

$$\begin{aligned} \dot{\eta}_v &= J(\eta)M^{-1}p_v^b \\ \dot{p}_v^b &= -J^\top(\eta)\frac{\partial P_v}{\partial \eta_v}(\eta_v) - \left[E^b(\eta, p^b) + D(M^{-1}p^b)\right]M^{-1}p_v^b + \tau_v. \end{aligned} \tag{6.43}$$

First step, consider the position dynamics in (6.43) with $y_{\tilde{\eta}_v} = \tilde{\eta}_v^b$ as output and p_v^b as "input". Define the control-like input as $p_v^b = \sigma_v^b + p_{vr}^b(\tilde{\eta}_v^b, t)$, where σ_v^b is a new input

[4]This ensures that the integral in (4.6) is well defined and independent of the path connecting 0 and $\tilde{q}_v$.

and $p^b_{vr}(\tilde{\eta}^b_v, t)$ as in (6.37). This results in the "closed-loop" position dynamics

$$\begin{cases} \dot{\eta}_v = J(\eta)M^{-1}p^b_d - \phi^b(\eta^b_v - \eta^b_d) + J(\eta)M^{-1}\sigma^b_v, \\ y_{\tilde{\eta}_v} = \eta_v - \eta^b_d(t), \end{cases} \tag{6.44}$$

whose prolonged system, in coordinates (6.36), is given by

$$\begin{cases} \dot{\tilde{\eta}}^b_v = -\phi^b(\tilde{\eta}^b_v) + J(\eta)M^{-1}\sigma^b_v \\ \delta\dot{\tilde{\eta}}^b_v = -\frac{\partial\phi^b}{\partial\tilde{\eta}^b_v}(\tilde{\eta}^b_v)\delta\tilde{\eta}^b_v + J(\eta)M^{-1}\delta\sigma^b_v \\ y_{\tilde{\eta}^b_v} = \tilde{\eta}^b_v, \\ \delta y_{\tilde{\eta}^b_v} = \delta\tilde{\eta}^b_v. \end{cases} \tag{6.45}$$

We consider as differential Lyapunov function the following

$$W_{\tilde{\eta}^b_v}(\tilde{\eta}^b_v, \delta\tilde{\eta}^b_v, t) = \frac{1}{2}\delta\tilde{\eta}^{b\top}_v \Pi^b_{\eta_v}(\tilde{\eta}^b_v, t)\delta\tilde{\eta}^b_v. \tag{6.46}$$

The time-derivative of (6.46) along solutions of (6.45) is

$$\begin{aligned} \dot{W}_{\tilde{\eta}^b_v} &= -\delta\tilde{\eta}^{b\top}_v \Pi^b_{\eta_v}(\tilde{\eta}^b_v, t)\frac{\partial\phi^b}{\partial\tilde{\eta}^b_v}(\tilde{\eta}^b_v)\delta\tilde{\eta}^b_v + \underbrace{\delta\tilde{\eta}^{b\top}_v \Pi^b_{\eta_v}(\tilde{\eta}^b_v, t)J(\eta)M^{-1}}_{\delta\overline{y}^\top_{\tilde{\eta}^b_v}}\delta\sigma^b_v + \frac{1}{2}\delta\tilde{\eta}^{b\top}_v \dot{\Pi}^b_{\eta_v}(\tilde{\eta}^b_v, t)\delta\tilde{\eta}^b_v, \\ &= \frac{1}{2}\delta\tilde{\eta}^{b\top}_v \left[\dot{\Pi}^b_{\eta_v}(\tilde{\eta}^b_v, t) - \Pi^b_{\eta_v}(\tilde{\eta}^b_v, t)\frac{\partial\phi^b}{\partial\tilde{\eta}^b_v} - \frac{\partial\phi^{b\top}}{\partial\tilde{\eta}^b_v}\Pi^b_{\eta_v}(\tilde{\eta}^b_v, t)\right]\delta\tilde{\eta}^b_v + \delta\overline{y}^\top_{\tilde{\eta}^b_v}\delta\sigma^b_v. \end{aligned} \tag{6.47}$$

By hypothesis (6.38), it follows immediately that

$$\dot{W}_{\tilde{\eta}^b_v}(\tilde{\eta}^b_v, \delta\tilde{\eta}^b_v, t) \leq -2\beta^b_{\eta_v}(\tilde{\eta}^b_v, t)W_{\tilde{\eta}^b_v}(\tilde{\eta}^b_v, \delta\tilde{\eta}^b_v, t) + \delta\overline{y}^\top_{\tilde{\eta}^b_v}\delta\sigma^b_v. \tag{6.48}$$

Hence system (6.44) is differentially passive with input-output pair $(\delta\sigma^b_v, \delta\overline{y}_{\tilde{\eta}^b_v})$ and differential storage function (6.46). This implies contraction when $\delta\sigma^b_v = 0$; furthermore, convergence to $\tilde{\eta}^b_v = 0$ if the input $\sigma^b_v = 0$ for system (6.45). This is imposed in next step.

Second step, consider (6.43) with output $y_{\sigma^b_v} = p^b_v - p^b_r(\tilde{\eta}^b_v, t)$. System (6.43) in coordinates (6.36) is expressed as the system composed of the $\tilde{\eta}^b_v$-dynamics in (6.45) and

$$\dot{\sigma}^b_v = -J^\top(\eta)\frac{\partial P_v}{\partial\eta_v}(\eta_v) - \left[E^b(\eta, p^b) + D(M^{-1}p^b)\right]M^{-1}p^b_v + \tau_v - \dot{p}^b_r, \tag{6.49}$$

where τ_v is given by (6.39). Substitution of $\tau_v^{ff}(x_v^b, x^b, t)$ in (6.40) yields

$$\dot{\sigma}_v^b = -\left[E^b(\eta, p^b) + D(M^{-1}p^b)\right] M^{-1}\sigma_v^b + \tau_v^{fb}. \tag{6.50}$$

With the above substitution, $\sigma_v^b = 0$ was imposed as a particular solution of (6.50). Thus, the prolongation of (6.43) in coordinates (6.36) is the system formed by (6.45) and

$$\begin{cases} \dot{\sigma}_v^b = -\left[E^b(\eta, p^b) + D(M^{-1}p^b)\right] M^{-1}\sigma_v^b + \tau_v^{fb}, \\ \delta\dot{\sigma}_v^b = -\left[E^b(\eta, p^b) + D(M^{-1}p^b)\right] M^{-1}\delta\sigma_v^b + \delta\tau_v^{fb}, \\ y_{\sigma_v^b} = \sigma_v^b, \\ \delta y_{\sigma_v^b} = \delta\sigma_v^b, \end{cases} \tag{6.51}$$

Now, let the candidate differential Lyapunov function for the prolonged system be given by (6.41) and consider the definition of the control action τ_v^{fb} in (6.40). Then,

$$\begin{aligned} \dot{W}_{x^b} &= \dot{W}_{\tilde{\eta}_v^b} + \delta\sigma_v^{b\top} M^{-1}\Big[-\left[E^b(\eta, p^b) + D(M^{-1}p^b)\right] M^{-1}\delta\sigma_v^b + \delta\tau_v^{fb}\Big], \\ &\leq -2\beta_{\eta_v}^b W_{\tilde{\eta}_v^b} + \delta\overline{y}_{\tilde{\eta}_v^b}^\top \delta\sigma_v^b + \delta\sigma_v^{b\top} M^{-1}\Big[-DM^{-1}\delta\sigma_v^b + \delta\tau_{fbv}\Big], \\ &= -2\beta_{\eta_v}^b W_{\tilde{\eta}_v^b} + \delta\overline{y}_{\tilde{\eta}_v^b}^\top \delta\sigma_v^b + \delta\sigma_v^{b\top} M^{-1}\Big[-DM^{-1}\delta\sigma_v^b - \Pi_{\eta_v}\delta\tilde{\eta}_v^b - K_d M^{-1}\delta\sigma_v^b + \delta\omega^b\Big], \\ &= -2\beta_{\eta_v}^b W_{\tilde{\eta}_v^b} + \delta\overline{y}_{\tilde{\eta}_v^b}^\top \delta\sigma_v^b + \underbrace{\delta\sigma_v^{b\top} M^{-1}}_{\delta\overline{y}_{\sigma_v^b}} \delta\omega^b - \delta\sigma_v^{b\top} M^{-1}(D + K_d)M^{-1}\delta\sigma_v^b - \delta\sigma_v^{b\top}\delta\overline{y}_{\tilde{\eta}_v^b}, \\ &\leq -2\min\{\beta_{\eta_v}^b, \lambda_{\min}\{D + K_d\}\lambda_{\min}\{M^{-1}\}\} W_{x^b} \delta\overline{y}_{\sigma_v^b}^\top \delta\omega^b. \end{aligned} \tag{6.52}$$

which completes the proof of the first item.

For the second part, similar to Remark 4.3, straightforward computations show that the closed-loop variational system is given by

$$\begin{bmatrix} \delta\dot{\tilde{\eta}}_v^b \\ \delta\dot{\sigma}_v^b \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{\partial\phi^b}{\partial\tilde{\eta}_v^b}(\tilde{\eta}_v^b)\Pi_{\eta_v}^{b-1}(\tilde{\eta}_v^b, t) & I \\ -I & -A^b(\eta, p^b) \end{bmatrix}}_{(\Xi^b - \Upsilon^b)} \underbrace{\frac{\partial^2 W}{\partial\tilde{x}_v^2}}_{\Pi^b} \begin{bmatrix} \delta\tilde{\eta}_v^b \\ \delta\sigma_v^b \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \delta\omega^b, \tag{6.53}$$

where $A^b(\eta, p^b) := \left[E(\eta, p^b) + D(M^{-1}p^b) + K_d\right]$, preserves the structure of (2.24) ■

Step 3: Original marine craft's tracking controller in the body frame

The design procedure in the inertial frame is finished by showing that the contraction based control scheme for the virtual system is also a tracking controller for the original marine craft in the body-fixed coordinates. This is stated in the following corollary.

Corollary 6.9. *Consider the controller* (6.39). *Then, all solutions of system* (6.19) *in closed-loop with the control law*

$$\tau(x^b, x^b, t) = \tau_v^{ff}(x^b, x^b, t) + \tau_v^{fb}(x^b, x^b, t)$$

converge exponentially to the trajectory $x_d^b(t)$ *with rate*

$$\beta^b = \min\{\beta^b_{\eta_v}, \lambda_{\min}\{D(M^{-1}p^b) + K_d\}\lambda_{\min}\{M^{-1}\}\}, \tag{6.54}$$

Proof: The results is proven by mimicking the steps of the proof in Corollary 4.4 ■

6.4.2 Control design in the inertial frame

Since the pH system (6.27) is already defined on the inertial frame $\{n\}$ and has also the structure of system (3.26); in the state (η, p) and matrix $E(\eta, p)$ as in (6.28). The v-CBC design procedure follows in the same manner as in Proposition 6.7. For this reason, the proofs are nod presented in this part.

Step 1: Virtual control system design

With the alternative form (6.27) in Proposition 6.7, we define a virtual system associated to the original pH system (6.24), in the state (η_v, p_v) and parametrized by (η, p), as the time-varying system

$$\begin{bmatrix} \dot{\eta}_v \\ \dot{p}_v \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & -(E_\eta(\eta, p) + D^H_\eta(\eta, p)) \end{bmatrix} \begin{bmatrix} \frac{\partial P_v}{\partial \eta_v}(\eta_v) \\ M_\eta^{-1}(\eta) p_v \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} \tau_{\eta v}, \tag{6.55}$$

This system also inherits the passivity property of (6.27) with storage function

$$H_{\eta_v}(\eta_v, p_v, t) = \frac{1}{2} p_v^\top M_\eta^{-1}(\eta) p_v + P_v(\eta_v), \tag{6.56}$$

for any $(\eta(t), p(t))$ and $t > t_0$, and supply rate $y_\eta^\top \tau_\eta$, where the output is $y_\eta = M_\eta^{-1}(\eta) p_v$.

Step 2: Contraction-based control design of the virtual pH system

Proposition 6.10. *Consider system* (6.55) *and a smooth trajectory* $x_d = (\eta_d, p_d) \in T^*Q$. *Let us introduce the following error coordinates*

$$\tilde{x}_v := \begin{bmatrix} \tilde{\eta}_v \\ \sigma_v \end{bmatrix} = \begin{bmatrix} \eta_v - \eta_d \\ p_v - p_r \end{bmatrix}, \tag{6.57}$$

and define the auxiliary momentum reference as

$$p_r(\tilde{\eta}_v, t) := M(\dot{\eta}_d - \phi(\tilde{\eta}_v)), \tag{6.58}$$

where $\phi : Q \to TQ$ *is such that* $\phi(0) = 0$; *and* $\Pi_{\eta_v} : Q \times \mathbb{R}_{\geq 0} \to \mathbb{R}^{n \times n}$ *is a positive definite Riemannian metric tensor satisfying the inequality*

$$\dot{\Pi}_{\eta_v}(\tilde{\eta}_v, t) - \Pi_{\eta_v}(\tilde{\eta}_v, t)\frac{\partial \phi}{\partial \tilde{\eta}_v}(\tilde{\eta}_v) - \frac{\partial \phi^\top}{\partial \tilde{\eta}_v}(\tilde{\eta}_v)\Pi_{\eta_v}(\tilde{\eta}_v, t) \leq -2\beta_{\eta_v}(\tilde{\eta}_v, t)\Pi_{\eta_v}(\tilde{\eta}_v, t), \tag{6.59}$$

with $\beta_{\eta_v}(\tilde{\eta}_v, t) > 0$, *uniformly. Assume also that the i-th row of* $\Pi_{\eta_v}(\tilde{\eta}_v, t)$ *is a conservative vector field. Consider also the composite control law given by*

$$\tau_{\eta v}(x_v, x, t) := \tau_{\eta v}^{ff}(x_v, x, t) + \tau_{\eta v}^{fb}(x_v, x, t) + \omega, \tag{6.60}$$

where $x = (\eta, p)$, $x_v = (\eta_v, p_v)$ *and*

$$\begin{aligned} \tau_{\eta v}^{ff} &= \dot{p}_r + \frac{\partial P_v}{\partial \eta_v} + [E_\eta(\eta, p) + D_\eta^H(\eta, p)]M_\eta^{-1}(\eta)p_r, \\ \tau_{\eta v}^{fb} &= -\int_0^{\tilde{\eta}_v} \Pi_{\eta_v}(\xi, t)d\xi - K_d M_\eta^{-1}(\eta)\sigma_v, \end{aligned} \tag{6.61}$$

where $K_d > 0$ *and* ω *is an external input. Then,*

1. *system* (6.55) *in closed-loop with* (6.60) *is differentially passive from* $\delta\omega$ *to* $\delta\overline{y}_{\sigma_v} = M_\eta^{-1}(\eta)\delta\sigma_v$ *with respect to the differential storage function*

$$W_x(\tilde{x}_v, \delta\tilde{x}_v, t) = \frac{1}{2}\delta\tilde{x}_v^\top \begin{bmatrix} \Pi_{\eta_v^b} & 0 \\ 0 & M_\eta^{-1}(\eta) \end{bmatrix} \delta\tilde{x}_v. \tag{6.62}$$

2. *the closed-loop variational dynamics preserves the structure of* (2.24)*, with*

$$\begin{aligned}\Pi(\tilde{x}_v, t) &= diag\{\Pi_{\eta_v}(\tilde{\eta}_v, t), M_\eta^{-1}(\eta)\} \qquad \Xi(\tilde{x}_v, t) = \begin{bmatrix} 0 & I \\ -I & -S_\eta^H(\eta, p) \end{bmatrix}. \\ \Upsilon(\tilde{x}_v, t) &= diag\left\{\frac{\partial \phi}{\partial \tilde{\eta}_v}(\tilde{\eta})\Pi_{\eta_v}^{-1}(\tilde{\eta}_v, t), D_\eta^H(\eta, p) + K_d\right\}, \qquad \Psi(\tilde{x}_v, t) = \begin{bmatrix} 0 \\ I \end{bmatrix}\end{aligned} \tag{6.63}$$

Proof: The proof follows exactly as in Proposition 4.2 ■

Step 3: Original marine craft's tracking controller in the inertial frame

Here we show that the controller that was designed previously can be used as a trajectoy tracking controller for the actual Hamiltonian system (6.24). This is stated as follows:

Corollary 6.11. *Consider the controller* (6.60)*. Then, all solutions of system* (6.19) *in closed-loop with the control law*

$$\tau_\eta(x, x, t) = \tau_{\eta v}^{ff}(x, x, t) + \tau_{\eta v}^{fb}(x, x, t)$$

converge exponentially to the trajectory $x_d(t)$ with rate

$$\beta = \min\{\beta_{\eta_v}, \lambda_{\min}\{D_\eta(\eta, p) + K_d\}\lambda_{\min}\{M_\eta^{-1}(\eta)\}\}. \tag{6.64}$$

Proof: It follows straightforward as in Corollary 4.4. ■

6.5 Example: Open-frame UUV

We consider the example in (Donaire et al. 2017) which is an open frame underwater vehicle with 140 kg of dry mass . The vehicle has four thrusters in an x-type configuration such that the system in fully-actuated in the degrees of freedom of interest, i.e., surge, sway and yaw-rate $\eta = (x, y, \psi)$ and $\nu = (u, v, r)$. The corresponding inertia, Coriolis and damping matrices in the body frame, respectively, are given by

$$M = \begin{bmatrix} 290 & 0 & 0 \\ 0 & 404 & 50 \\ 0 & 50 & 132 \end{bmatrix}, \qquad C(\nu) = \begin{bmatrix} 0 & 0 & -404v - 50r \\ 0 & 0 & 290u \\ 404v + 50r & -290u & 0 \end{bmatrix},$$

$$D(\nu) = \begin{bmatrix} 95 + 268|v| & 0 & 0 \\ 0 & 613 + 164|u| & 0 \\ 0 & 0 & 105 \end{bmatrix}.$$

6.5.1 Tracking in the body-fixed frame

For the simulation in the body frame, consider the following set-u: $\phi^b(\tilde{\eta}_v^b) = \Lambda\tilde{\eta}_v^b$ where $\Lambda = \text{diag}\{0.6, 0.8, 0.2\}$, $\Pi_{\eta_v}^b = \Lambda$ and $K_d = \text{diag}\{300, 100, 200\}$. The time performance of system generalized position η and the reference trajectory

$$\eta_d(t) = (3\sin(t), 3\cos(t), \arctan(\sin(t), \cos(t))) \tag{6.65}$$

with initial conditions $\eta = (1, -1, 40\pi/180)$ and $\nu = (0, 0, 0)$ is shown in Figure 6.3.

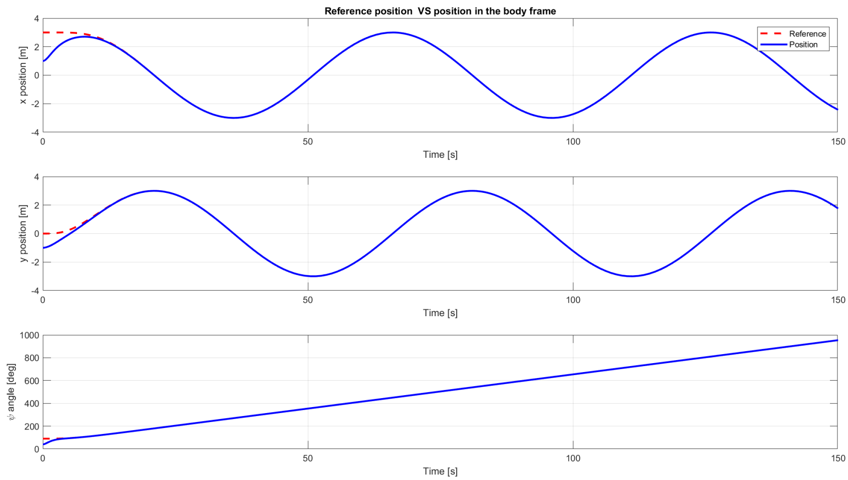

Figure 6.3: Vehicle's position vector η against η_d.

After a short transient, the system's position tracks asymptotically the reference trajectory η_d. This can be better appreciated in the time response of the error signals $\tilde{\eta}$ and $\tilde{\sigma}^b$ in Figure 6.4, where after a short transitory time, all signals converge to zero.

Similarly, in Figure 6.40 the time response of the control signals are shown. The overshoot in the control signals is a consequence of the high-gain that is used to perform the task fast enough, which can be alleviated by retuning the gains or considering a saturated controller similar to the control scheme (4.55).

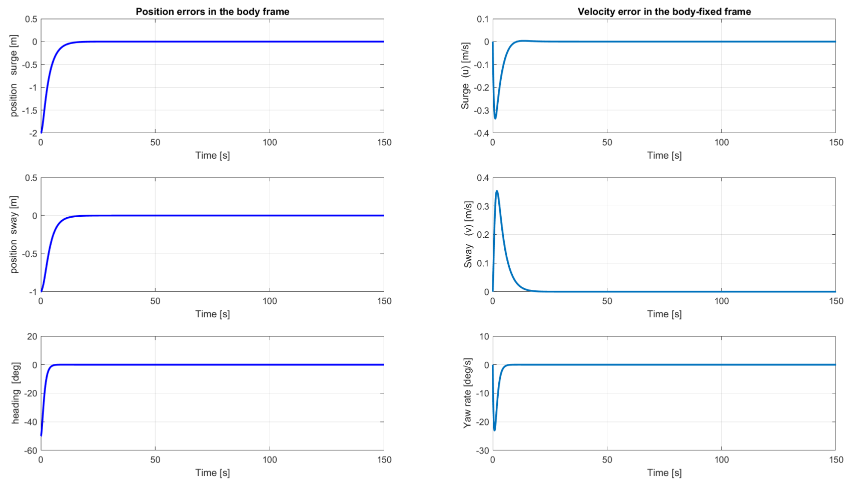

Figure 6.4: Vehicle's position and pseudo-velocity errors in the body-frame.

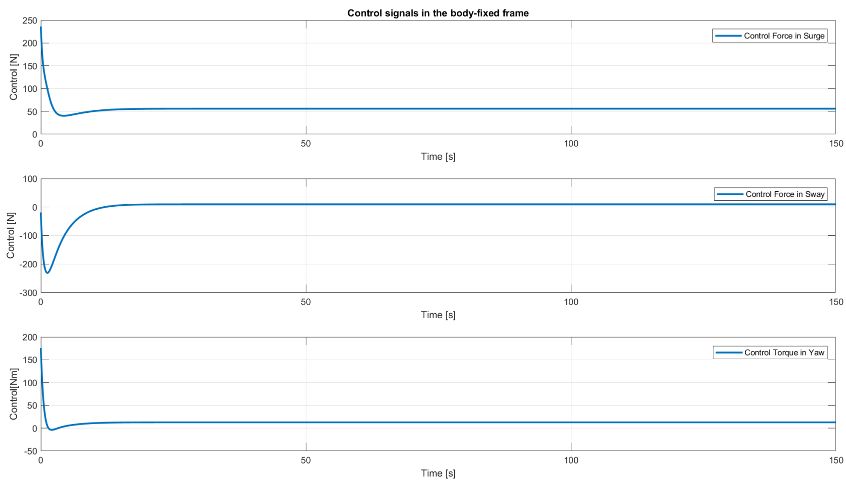

Figure 6.5: Control signal in the body frame.

The desired trajectory in the Cartesian space versus the marine craft's Cartesian position is shown in Figure (6.6). The marine craft tracks a circle in a smooth manner.

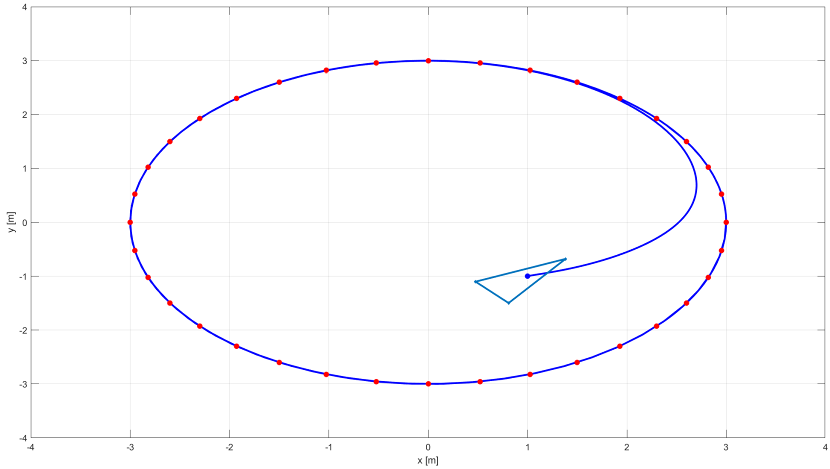

Figure 6.6: Desired trajectory in the Cartesian space.

6.5.2 Tracking in the inertial frame

For the simulation in the inertial frame, consider the following set-u: $\phi^b(\tilde{\eta}_v^b) = \Lambda\tilde{\eta}_v$ in (6.61), where the same trajectory (6.65) and gains of the previous example are taken. Position and attitude and its rate of change are the same as in the body-fixed. However, the pseudo-velocities projected in the inertial frame are true velocities, see (Greenwood 2003). Thus, in the inertial case, only the plots of the true velocity error and control signal in the inertial frame are shown in Figure 6.7 and Figure 6.8.

6.6 Conclusions

In this chapter v-CBC the method has been applied to solve the trajectory tracking problem in a fully-actuated marine craft. Similar to the previous chapters, the pH structure and associated workless forces have been exploited to construct virtual marine craft models. The exponential convergence to a unique predefined steady-state trajectory is guaranteed by the contractivity property the virtual system. Two families of control schemes based on body-fixed attitude and velocity measurements were developed, one for the pH model in the body-fixed frame $\{b\}$, and another for the pH model in the inertial frame $\{n\}$. The structure of the pH models and associated virtual systems suggest that these models are independent of the coordinate frame. Simulations confirm the expected performance of the closed-loop system.

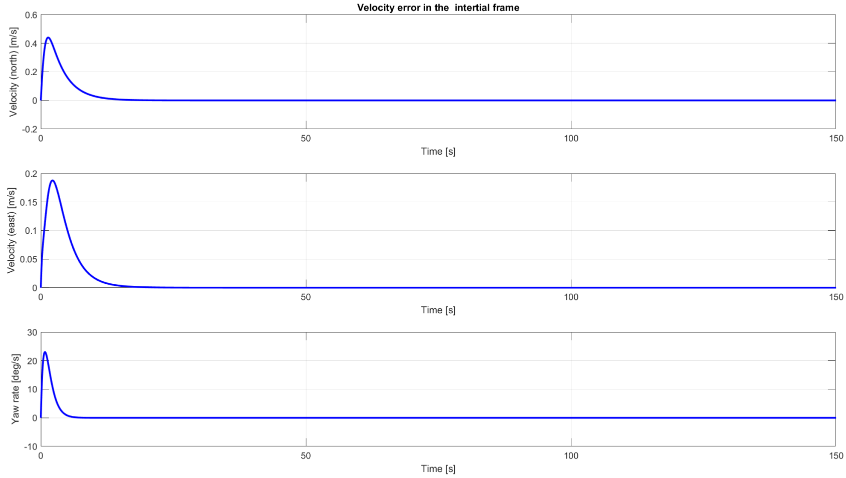

Figure 6.7: Velocity error signals in the inertial frame.

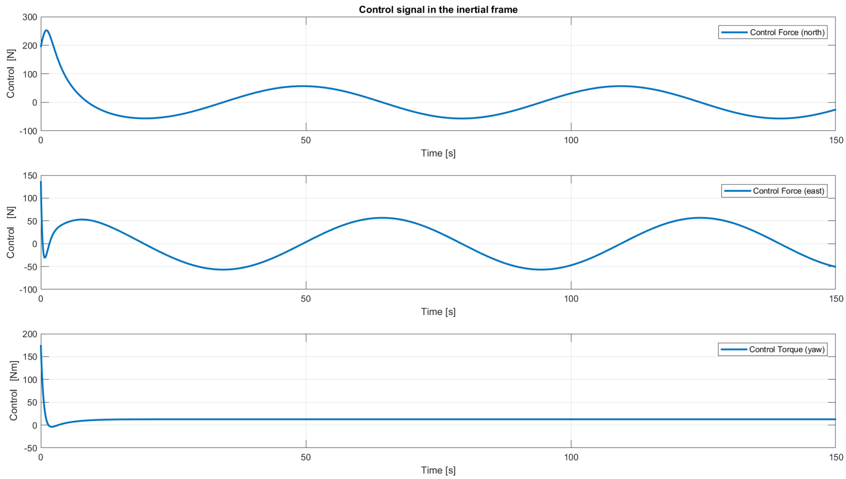

Figure 6.8: Velocity error signals in the inertial frame.

Both models can be used in the design process of a tracking control scheme of marine craft. In particular, for the implementation it should be considered that attitude and velocities sensors (like the IMU) measure in the body-fixed coordinates; whereas the GPS or cameras measure in the inertial frame. Thus, by employing both models, it is possible to understand the effects of the proportional gain (accting in the inertial frame coordinates), and the derivative gain (in the body frame coordinates).

Some problems are left out open. For instance, the extension of the above approach to underactuated marine craft. Secondly, in the control of marine systems literature, a common practice is to develop the technique in the context of path-following (maneuvering regulation); thus, the above approach is expected to be extended in this setting. Finally, we will exploit the geometry of the configuration space to formulate the control problem on the *Special Euclidean group* $SE(3)$ in order to understand the intrinsic properties of the approach.

Chapter 7

Passivity-based distributed control of networked Euler-Lagrange systems

”However beautiful the strategy, you should occasionally look at the results.”

- Sir Winston Churchill

In this chapter the problem of distributed coordination tracking control is solved by means of the passivity properties of virtual systems in the Euler-Lagrange framework. To this end, the passivity-based control design method in (Arcak 2007) is reformulated by considering that each edge is associated with an *artificial spring dynamical system* instead of the usual diffusive coupling among the communicating agents. With this configuration, the networked EL system possesses a ”symmetric” feedback structure which together with the strict passivity of both agents’ and edges’ dynamics lead to a strictly passive (virtual) network dynamics. Subsequently, it is shown that the networked version of two different passivity-based tracking controllers in the literature are particular cases of the proposed technique. Numerical simulations are presented to show the performance of the proposed method.

7.1 Introduction

The generalization of these PBC methods to the multi-agent setting has been well-studied in recent decade. The books of (Bai et al. 2011), (van der Schaft 2017) and the articles by (Chopra and Spong 2006) and (Arcak 2007) provide a thorough exposition to the design of passivity-based distributed control where a number of coordination control problems can be solved through PBC approach, including, synchronization and formation control. For networked EL systems, some relevant works are the articles by (Garcia de Marina Peinado et al. 2018), (Nuño, Ortega, Jayawardhana and Basanez 2013), (Nuño, Ortega, Jayawardhana and Basañez 2013) and (Chung and Slotine 2009a). In the case of pH systems, the most relevant work is presented in (Vos 2015), where the coordination problem for point masses in the pH framework was solved.

In the chapter, the PBC design methodology in (Arcak 2007) is reformulated. The coordination control problem is solved first by interconnecting (strictly passive) systems attached to the nodes of a graph via diffusive coupling that preserves the passivity of the network dynamics. As an alternative, we attach strictly passive *artificial spring systems* to each node and they are feedback interconnected to the nodes dynamics. The later results in dynamic coordination protocols where the spring dynamics can be interpreted as a (nonlinear) integral action. Due to the strict passivity of the interconnected system, the asymptotic stability result can be established by using the total storage function as a *strict* Lyapunov function.

7.2 Networked Euler-Lagrange systems preliminaries

In this work we consider a network of N EL systems (agents) which interact among themselves for solving tracking control problem in a *coordinated manner*. The interaction among the agents in the network is represented by the edges of a graph.

7.2.1 A prime on graph theory

The following preliminaries of graph theory are taken from (Bollobás 1998) and (Van der Schaft and Maschke 2013).

Definition 7.1. *A graph $\mathcal{G}$ is defined by a pair $(\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is a finite set of N vertices (also called nodes) and $\mathcal{E}$ the finite set of M edges (also called links). Furthermore, there is an injective mapping from $\mathcal{E}$ to the set of unordered pairs $\mathcal{V}$, identifying edges with unordered pairs of vertices.*

The set $\mathcal{V}$ is called the vertex set of the graph $\mathcal{G}$ and the set $\mathcal{E}$ is called the edge set. The graph $(\mathcal{V}, \mathcal{E})$ is said to be *directed* if the edges are *ordered* pairs $e_{ij} = (w_i, w_j)$. Then w_i represents the tail vertex and w_j the head vertex of e_{ij}. If $e_{ij} = (w_i, w_j) \in \mathcal{E}$, then the vertices w_i and w_j are called *adjacent* or *neighboring* vertices; and w_i and w_j are *incident* with the edge e_{ij}. Two edges are *adjacent* if they have exactly one common end-vertex. A *path* (weak path) of length r in a directed graph is a sequence $w_1, \ldots, w_\iota$ of $\iota+1$ distinct vertices such that for each $i \in \{1, \ldots, \iota\}$, $(w_i, w_{i+1}) \in \mathcal{E}$ (respectively, either $(w_i, w_{i+1}) \in \mathcal{E}$ or $(w_{i+1}, w_i) \in \mathcal{E}$). A directed graph is *strongly connected* (weakly connected) if any two vertices can be joined by a path (respectively, weak path).

Given a graph $\mathcal{G}$, we define its *vertex space* $\Lambda_0(\mathcal{G})$ as the vector space of all functions from $\mathcal{V}$ to some linear space $\mathcal{R}$. Furthermore, we define the *edge space* $\Lambda_1(\mathcal{G})$ as the vector space of all functions from $\mathcal{E}$ to $\mathcal{R}$. The dual spaces of Λ_0 and Λ_1 will be denoted by Λ^0 and Λ^1, respectively. For a directed graph $\mathcal{G}$, the *incidence operator* is a linear map $B : \Lambda_1 \to \Lambda_0$ with matrix representation $\hat{B} \otimes I$, where

$$\hat{B} \otimes I := \begin{bmatrix} \hat{b}_{11}I & \dots & \hat{b}_{1M}I \\ \vdots & \ddots & \vdots \\ \hat{b}_{N1}I & \dots & \hat{b}_{NM}I \end{bmatrix} \tag{7.1}$$

where $\otimes$ denotes de Kronecker product, I be the matrix representation of the identity map $I : \mathcal{R} \to \mathcal{R}$ of appropriate dimensions, and

$$\hat{b}_{ik} := \begin{cases} -1 & \text{if node } i \text{ is at the tail of } k\text{-th edge} \\ 1 & \text{if node } i \text{ is at the head of } k\text{-th edge} \\ 0 & \text{otherwise.} \end{cases} \tag{7.2}$$

is the *incidence matrix*. The adjoint operator of B is given by the map $B^* : \Lambda^0 \to \Lambda^1$ with matrix representation $\hat{B}^\top \otimes I$, and it is called the *co-incidence operator*. We will throughout use B ($B^\top$) for both the incidence (respectively, co-incidence) matrix and for incidence (respectively, co-incidence) operator. The rank of B is at most $N-1$ due to the sum of its rows is zero. Indeed, the rank is $N-1$ when the graph is connected. The columns of B are linearly independent. We also introduce the *Laplacian matrix* given by

$$L := BB^\top. \tag{7.3}$$

7.2.2 Euler-Lagrange network dynamics

Consider agents evolving on a configuration manifold $\mathcal{Q}_i$ of dimension n for all $i \in \{1, \dots, N\}$. The position of the i-th robot is given by the vector $q_i \in \mathcal{Q}_i$ and velocity $\dot{q}_i \in T_{q_i}\mathcal{Q}_i$. The dynamics of the i-th agent are given by the EL equations in (3.1), which for sake of completeness are presented also here and is assumed that $\operatorname{rank}(B(q)) = n$,

$$\begin{aligned} \dot{q}_i &= v_i, \\ M_i(q_i)\dot{v}_i + C_i(q_i, v_i)v_i + g_i(q_i) &= \tau_i, \end{aligned} \tag{7.4}$$

The total energy of system (7.4) is given by

$$\mathcal{E}_i(q_i, v_i) = \frac{1}{2}\dot{q}_i^\top M_i(q_i)\dot{q}_i + P_i(q_i). \tag{7.5}$$

Remark 7.2. *It is not assumed that all the agents are identical.*

The EL network dynamics can be expressed in a compact form by using,

$$\boldsymbol{q} := [q_1^\top, \ldots, q_N^\top]^\top, \quad \boldsymbol{v} := [v_1^\top, \ldots, v_N^\top]^\top, \boldsymbol{\tau} := [\tau_1^\top, \ldots, \tau_N^\top]^\top, \quad \boldsymbol{g} := [g_1^\top, \ldots, g_N^\top]^\top,$$

together with the block diagonal matrices

$$\boldsymbol{M}(\boldsymbol{q}) := \mathrm{diag}\{M_1(q_1), \ldots, M_N(q_N)\}, \quad \boldsymbol{C}(\boldsymbol{q}, \boldsymbol{v}) := \mathrm{diag}\{C_1(q_1, v_1), \ldots, C_N(q_N, v_N)\}.$$

Thus, the network dynamics with state (q, v) is given by

$$\begin{aligned} \dot{\boldsymbol{q}} &= \boldsymbol{v}, \\ \boldsymbol{M}(\boldsymbol{q})\dot{\boldsymbol{v}} + \boldsymbol{C}(\boldsymbol{q}, \boldsymbol{v})\boldsymbol{v} + \boldsymbol{g}(\boldsymbol{q}) &= \boldsymbol{\tau}, \end{aligned} \tag{7.6}$$

where, $\boldsymbol{g}(\boldsymbol{q}) = \frac{\partial P_T}{\partial \boldsymbol{q}}(\boldsymbol{q})$ with the network's potential function given by

$$P_T(\boldsymbol{q}) := \sum_{i=1}^{N} P_i(q_i). \tag{7.7}$$

Notice that network dynamics (7.6) preserves the structure of the i-th agent in (7.4) dynamics. For instance, the matrix

$$\boldsymbol{N}(\boldsymbol{q}, \boldsymbol{v}) := \dot{\boldsymbol{M}}(\boldsymbol{q}) - 2\boldsymbol{C}(\boldsymbol{q}, \boldsymbol{v}) \tag{7.8}$$

is skew-symmetric due the for each agent the matrix $N_i(q_i, v_i) = \dot{M}_i(q_i) - 2C_i(q_i, v_i)$ is skew-symmetric, see Section B.1.2. Also the map $\tau \mapsto v$ is lossless with storage function

$$\mathcal{E}(\boldsymbol{q}, \boldsymbol{v}) = \sum_{i=1}^{N} \left(\frac{1}{2} v_i^\top M_i(q_i) v_i + P_i(q) \right) = \frac{1}{2} \boldsymbol{v}^\top \boldsymbol{M}(\boldsymbol{q}) \boldsymbol{v} + P_T(\boldsymbol{q}). \tag{7.9}$$

7.2.3 Passivity based tracking controllers for a single agent

In this section, two well known passivity based tracking controllers of mechanical systems are recalled. The first one in the celebrated work of (Slotine and Li 1987), and the second one is a backstepping re-design in (Fossen and Berge 1997) of the first. In these two schemes the passivity preserving property of the virtual system (3.8) is key for the closed-loop exponential convergence proof as it will be sketched bellow.

Consider the virtual system of to the i-th EL system (7.4) given by[1]

$$\begin{aligned}\dot{q}_{iv} &= v_{vi},\\ M_i(q_i)\dot{v}_{vi} + C_i(q_i, v_i)v_{vi} + g_{vi}(q_{vi}) &= \tau_{vi},\\ y_{vi} &= v_{vi},\end{aligned} \tag{7.10}$$

where the map $\tau_{vi} \mapsto v_{vi}$ is lossless with storage function (see Proposition 3.11)

$$\mathcal{E}_{vi}(q_{vi}, v_{vi}, q_i) = \frac{1}{2}v_{vi}^\top M_i(q_i)v_{vi} + P_{vi}(q_{vi}), \tag{7.11}$$

parametrized by q_i. With the preliminary feedback

$$\tau_{vi} = g_{vi}(q_{vi}) - K_i v_{vi} + u_{vi}, \tag{7.12}$$

with $K_i > 0$ and v_{vi} an external input, the position and velocity dynamics are decoupled. This yield the velocity dynamics

$$\begin{aligned}M_i(q_i)\dot{v}_{vi} + C_i(q_i, v_i)v_{vi} + K_i v_{vi} &= u_{vi},\\ y_{vi} &= v_{vi},\end{aligned} \tag{7.13}$$

which is output strictly passive (see (van der Schaft 2017)) with storage function

$$\mathcal{S}_{vi}^{SL}(v_{vi}, q_i) = \frac{1}{2}v_{vi}^\top M_i(q_i)v_{vi}, \tag{7.14}$$

parametrized by q_i. Indeed, the time derivative of (7.14) is

$$\begin{aligned}\dot{\mathcal{S}}_{vi}^{SL}(v_{vi}, q_i) &= \frac{1}{2}v_{vi}^\top \dot{M}_i(q_i)v_{vi} + v_{vi}^\top\left(u_{vi} - C_i(q_i, v_i)v_{vi} - K_i v_{vi}\right),\\ &= v_{vi}^\top u_{vi} + v_{vi}^\top N_i(q_i, v_i)v_{vi}^\top - v_{vi}^\top K_i v_{vi}^\top,\\ &= y_i^\top u_i - v_{vi}^\top K_i v_{vi}^\top.\end{aligned} \tag{7.15}$$

From a design point of view, this fact makes (7.13) also a suitable of target closed-loop system. In fact, this is one of the main features in (Slotine and Li 1987), where a *sliding manifold* (see Definition 4.39) is made invariant and attractive by designing the *sliding (error) variable*; v_{vi} in this case; such that it satisfies the velocity dynamics in (7.13). This idea is closely related to the one in Corollary 4.14, but invoking passivity arguments instead of virtual contraction.

Theorem 7.3 (Slotine-Li controller)**.** *Consider a smooth reference trajectory $q_d(t)$ to-*

[1] In (Slotine and Li 1987) and (Fossen and Berge 1997) the virtual velocity v_{vi} is denoted with a letter s.

gether with the change of variables $v_{vi} = v_i - v_{ri}$ *where* $v_{ri} = \dot{q}_d - \Pi_i(q_i - q_d)$ *is an artificial velocity reference with* $\Pi_i = \Pi_i^\top > 0$. *Then, the EL system* (7.4) *in closed-loop with the control law*

$$\tau_i = M_i(q_i)\dot{v}_{ri} + C_i(q_i, v_i)v_{ri} + g_i(q_i) - K_i v_{vi} + u_{vi} \tag{7.16}$$

is given by the virtual system (7.13) *and* $v_{vi}(t) \to 0_n$, $\dot{q}(t) - \dot{q}_d(t) \to 0$ *and* $q(t) - q_d(t) \to 0$.

This control law gives exponential stability as shown in (Spong et al. 1990) via a strict Lyapunov function. Due to the rectangular form of system (7.10), in (Fossen and Berge 1997) a *backstepping redesign* is proposed, where crossed terms in the closed-loop dynamics are incorporated.

Theorem 7.4 (Vectorial backstepping)**.** *Consider the change of coordinates* $\tilde{q}_i = q_i - q_d$ *and all hypothesis of Theorem 7.3. Then, the EL system* (7.4) *in closed-loop with the controller*

$$\tau_i = M_i(q_i)\dot{v}_{ri} + C_i(q_i, v_i)v_{i,ri} + g_i(q_i) - K_i v_{vi} - \Pi_i\tilde{q}_i + u_{vi} \tag{7.17}$$

is given by

$$\begin{aligned} \dot{\tilde{q}}_i + \Pi_i\tilde{q}_i &= v_{vi}, \\ M_i(q_i)\dot{v}_{vi} + C_i(q_i, v_i)v_{vi} + K_i v_{vi} + \Pi_i\tilde{q}_i &= u_{vi}, \\ y_{vi} &= v_{vi}, \end{aligned} \tag{7.18}$$

and the origin $(\tilde{q}_i, v_{vi}) = (0, 0)$ *is a globally uniformly exponentially stable equilibrium point.*

A sketch of the proof is as follows: the map $u_{vi} \mapsto v_{vi}$ is strictly passive for closed-loop system (7.18) with respect to the storage function

$$\mathcal{S}_{vi}^{bk}(\tilde{q}_i, v_{vi}, q_i) = \frac{1}{2}v_{vi}^\top M_i(q_i)v_{vi} + \frac{1}{2}\tilde{q}_i^\top\Pi_i\tilde{q}_i. \tag{7.19}$$

This in turn implies that $\mathcal{S}_{vi}(\tilde{q}_i, v_{vi}, q_i)$ is a strict Lyapunov function by fixing $u_{vi} = 0_n$.

Remark 7.5. *The schemes in Theorem 7.3 and Theorem 7.4 can be generalized to a class of nonlinear passivity based controllers characterized by Corollary 4.9 in terms of incrementally passive maps.*

7.3 Distributed node & edge dynamic controller design

7.3.1 Group coordination problem formulation

The problem is formulated as follows: Design a distributed control law for the network of EL systems (7.6) on each node and edge based only on local information. That is, the i-th agent can use the information $y_i - y_j$ if the j-th agent is a neighbor, where y_i is the *passive output* of (7.4). To this end, the following behavior must be guaranteed:

- Each agent's velocity $v_i(t)$ in (7.4) tracks a the artificial velocity reference $v_{ri}(t)$ in (7.16), prescribed for each agent in the network; that is

$$\lim_{t\to\infty} \|v_i(t) - v_{ri}(t)\| = 0 \quad \text{for} \quad i \in \{1, \dots, N\}, \tag{7.20}$$

  where q_d is common to all the agents in the network.

- The interaction variable ζ_k (defined on the edge and denotes the relative displacement between agents i and j interconnected by an *artificial spring* that will be defined shortly below) *converges to a nonempty compact set* $\mathcal{A}_k \subset \mathcal{Q}$, for all $k \in \{1, \dots, M\}$.

Remark 7.6. *Only $\mathcal{A}_k = \{\zeta_d\}$ is considered here, with ζ_d is a desired relative displacement.*

7.3.2 Node & edge dynamic control design method

The proposed design procedure is as follows:

1. For each EL agent (7.4), consider the nodal feedback control (7.17) (respectively (7.16)) such that the closed-loop system is passive from an external input u_{vi} to the velocity "error" $y_i := v_{vi} = v_i - v_r$. Denote this local closed-loop system as $y_i = \mathcal{H}_i\{u_{vi}\}$.

2. For every edge k, assign a (spring) system

$$\Sigma^{\zeta}_{\mu k} : \begin{cases} \dot{\zeta}_k = \mu_k - \phi_{\zeta k}(\zeta_k), \\ \tau^{\zeta}_k = \frac{\partial P_{\zeta k}}{\partial \zeta_k}(\zeta_k), \quad k \in \{1, \dots, M\}, \end{cases} \tag{7.21}$$

   to the k-th edge if $(i, j) \in \mathcal{E}_k$, where $\phi_{\zeta k}$ is a potential force to be designed and $P_{\zeta k} : \mathcal{D}_k \subseteq \mathcal{Q}_k \to \mathbb{R}_{\geq 0}$ is a C^2 convex artificial potential energy function defined on the open set $\mathcal{D}_k$ with minimum at ζ_d where ζ_d is the vector of desired relative displacement between communicating agents.

3. Interconnect all node systems $y_i = \mathcal{H}_i\{u_i\}$ and all edge spring system $\Sigma^{\zeta}_{\mu k}$ through the following passivity preserving interconnection

$$\mu_k := \sum_{\ell=1}^{N} b_{\ell k} y_\ell, \qquad u_{vi} = -\sum_{k=1}^{M} b_{ik}\tau^{\zeta}_k, \tag{7.22}$$

for $k \in \{1, \dots, M\}$, and $i \in \{1, \dots, N\}$, respectively.

The function $P_{\zeta k}(\zeta_k)$ is designed to render the target sets $\mathcal{A}_k$ invariant and asymptotically stable for $k \in \{1, \dots, M\}$. Interconnection laws in (7.22) satisfy the vector relations

$$\mu = (B^\top \otimes I_n)y \quad \text{and} \quad u_v = -(B \otimes I_n)\tau, \tag{7.23}$$

with $y = [y_1^\top, \dots, y_N^\top]^\top$, and $\mu = [\mu_1^\top, \dots, \mu_M^\top]^\top$. This means that μ must lie in $\mathrm{Im}(B^\top \otimes I_n)$. Hence, for the second design goal to be feasible, $\mathcal{A}_k$ must be nonempty, $\forall k \in \{1, \dots, M\}$.

Note that, in this case, we assume that $\mathcal{A}_k = \{\zeta_d\}$ and $P_{\zeta k}$ satisfying:

- $P_{\zeta k}(\zeta_k) \to \infty$ as $\zeta_k \to \partial\mathcal{D}^\infty_k$,
- $P_{\zeta k}(\zeta_k) = 0 \iff \zeta_k \in \mathcal{A}_k$,
- $\frac{\partial P_{\zeta k}}{\partial \zeta_k}(\zeta_k) = 0 \iff \zeta_k \in \mathcal{A}_k$.

The notation $\zeta_k \to \partial\mathcal{D}^\infty_k$ indicate that ζ_k converges to the boundary of $\mathcal{D}_k$ as $\|\zeta_k\| \to \infty$.

Remark 7.7. *Some observations with respect to the design procedure in (Arcak 2007).*

- *If $\phi_{\zeta k}(\zeta) = 0$ in* (7.23)*, then the procedure in (Arcak 2007) recovered.*
- *In this work the feasibility condition*

$$(\mathcal{A}_1 \times \cdots \times \mathcal{A}_M) \cap \mathcal{R}a(B^\top \otimes I_n) \neq \emptyset.$$

  is not required, However, $\mathcal{A}_k \neq \emptyset$, for all $k \in \{1, \dots, M\}$, which is set up by design.

- *Due to relations in* (7.23)*, the "symmetric" interconnection structure is preserved.*

For the first step of the group coordination problem, the PBC schemes in Theorem 7.3 or Theorem 7.4 are used as internal feedback. The corresponding passive map for the i-th agent is denoted by $y_i = \mathcal{H}_i\{u_i\}$.
With the introduction of the concatenated vectors

$$y := [y_1^\top, \dots, y_N^\top]^\top, \quad \mathcal{H} := [\mathcal{H}_1^\top, \dots, \mathcal{H}_N^\top]^\top, \quad u := [u_1^\top, \dots, u_N^\top]^\top, \quad \psi := [\psi_1^\top, \dots, \psi_M^\top]^\top,$$

the passive map $u_i \mapsto y_i$ for $i \in \{1, \ldots, N\}$ and external control (7.22); with $\tau_k^\zeta = \psi_k(\zeta_k)$ for $k \in \{1, \ldots, M\}$; is expressed in compact from as

$$y = \mathcal{H}\{u_v\} \quad \text{and} \quad u_v = -(B \otimes I_n)\psi(\zeta). \tag{7.24}$$

In (Arcak 2007) it was shown that the interconnection between $\mathcal{H}\{u\}$ and $\psi(\zeta)$ exhibits a "symmetric" feedback interconnection structure due to multiplication by the matrices $(B \otimes I_n)$ and $(B^\top \otimes I_n)$ which preserves passivity.

Similar to the first step in the design procedure in (Arcak 2007), it also imposed that the system $y_i = \mathcal{H}_i\{u_i\}$ to be *strictly passive* from u_i to y_i with a C^1, positive definite, radially unbounded storage function (7.14) or (7.19) satisfying

$$\mathcal{S}_{vi}^{SL}(v_{vi}, q_i) \leq W_{y,i}(v_{vi}) + y_i^\top u_{vi}, \quad \text{or} \quad \mathcal{S}_{vi}^{bk}(\tilde{q}_i, v_{vi}, q_i) \leq W_{y,i}(v_{vi}) + y_i^\top u_{vi}, \tag{7.25}$$

for the positive definite function $W_{yi}(v_{vi}) = v_{vi}^\top K v_{vi}$, with $K > 0$.

Likewise, for the spring system we also require that its dynamics (7.21) to be strictly passive. To this end, take[2] $\phi_{\zeta k}(\zeta_k) = \tau_k^\zeta$ in (7.21). It follows that the map $\mu_k \mapsto \tau_k$ is strictly passive with $P_{\zeta,k}(\zeta_k)$ as storage function.
This is easily seen from the time derivative of $P_{\zeta,k}(\zeta_k)$ along system (7.21),

$$\dot{P}_{\zeta k} = \frac{\partial P_{\zeta k}^\top}{\partial \zeta_k}(\zeta)\mu_k - \underbrace{\frac{\partial P_{\zeta k}^\top}{\partial \zeta_k}(\zeta)\frac{\partial P_{\zeta k}}{\partial \zeta_k}(\zeta)}_{W_{\zeta k}(\zeta_k)}. \tag{7.26}$$

If $W_{\zeta,k}(\zeta) = 0$, then the passive map $\mu_k \mapsto \tau_k$ is a lossless. Due to convergence purposes only the first case is considered in this work. In this work the following choice is made:

$$P_{\zeta k}(\zeta_k) = \frac{1}{2}(\zeta_k - \zeta_d)^\top K_\zeta(\zeta_k - \zeta_d). \tag{7.27}$$

7.3.3 Interconnected system stability analysis

With the previous established set up, the following proposition is stated.

Proposition 7.8. *Consider system $\mathcal{H}_i\{u_i\}$, $i \in \{1, \ldots, N\}$ and system $\Sigma_{\mu k}^\zeta$ in* (7.21)*, $k \in \{1, \ldots, M\}$ and take $\mathcal{A}_k = \{\zeta_d\}$ for all $k \in \{1, \ldots, M\}$. Then, the set*

$$C = \{(\zeta, y) | y = 0 \quad \textit{and} \quad z \in \mathcal{A}_1 \times \cdots \times \mathcal{A}_M\}. \tag{7.28}$$

is uniformly asymptotically stable.

[2]Different options are possible. A class of functions $P_{\zeta k}$ can be characterized by Definition 4.32.

Proof: Take as Lyapunov function to

$$V(\zeta, y) = \sum_{k=1}^{M} S_i(\tilde{q}_i, v_{vi}) + \sum_{k=1}^{M} P_{\zeta k}(\zeta_k), \tag{7.29}$$

where $S_i(\tilde{q}_i, v_{vi})$ is either (7.14) or (7.19). Theorem 7.3 (or Theorem 7.4) and (7.26) imply that $\dot{V}(\zeta, y)$ is a strict Lyapunov function for the set C. ■

With above preliminaries, the main result is stated as follows:

Proposition 7.9. *Consider agent's dynamics* (7.4) *which is in closed-loop with the control law* (7.17) *for all* $i \in \{1, \dots, N\}$*, combined with* (7.21) *and* (7.23) *and* $P_{\zeta k}(\zeta_k)$ *as in* (7.27). *Then, the point* $(\tilde{q}, v_v, \zeta) = (0, 0, \zeta_d)$ *is uniformly exponentially stable with rate* $\beta = k_3/k_2$*, where*

$$\begin{aligned} k_2 &= \max\{\lambda_{\max}(\Pi), \lambda_{\max}(M(q)), \lambda_{\max}(K_\zeta)\}, \\ k_3 &= \min\{\lambda_{min}(\Pi^2), \lambda_{\min}(K), \lambda_{\min}(K_\zeta^2)\}. \end{aligned} \tag{7.30}$$

with $\Pi := diag\{\Pi_1, \dots, \Pi_N\}$ *and* $K = diag\{K_1, \dots, K_N\}$.

Proof: First notice that the closed-loop system is

$$\begin{aligned} \dot{\tilde{q}} + \Pi\tilde{q} &= v_v, \\ M(q)\dot{v}_v + C(q, v)v_v + Kv_v &= -\Pi\tilde{q} - (B \otimes I_n)\frac{\partial P_\zeta}{\partial \zeta}(\zeta), \\ \dot{\zeta} + \frac{\partial P_\zeta}{\partial \zeta}(\zeta) &= (B^\top \otimes I_n)v_v \end{aligned} \tag{7.31}$$

Take as a candidate Lyapunov function

$$S_{\tilde{q}v_v\zeta}(\tilde{q}, v_v, \zeta, q) = \frac{1}{2}\begin{bmatrix} \tilde{q} \\ v_v \end{bmatrix}^\top \begin{bmatrix} \Pi & 0 \\ 0 & M(q) \end{bmatrix} \begin{bmatrix} \tilde{q} \\ v_v \end{bmatrix} + \sum_{k=1}^{M} P_{\zeta k}(\zeta_k), \tag{7.32}$$

which satisfies the bounds (see (Khalil 1996))

$$k_1 \left\| \begin{bmatrix} \tilde{q} \\ v_v \\ \zeta - \zeta_d \end{bmatrix} \right\|^2 \le S_{\tilde{q}v_v\zeta}(\tilde{q}, v_v, \zeta, q) \le k_2 \left\| \begin{bmatrix} \tilde{q} \\ v_v \\ \zeta - \zeta_d \end{bmatrix} \right\|^2 \tag{7.33}$$

where

$$k_1 = \min\{\lambda_{min}(\Pi), \lambda_{min}(M(q)), \lambda_{min}(K_\zeta)\}.$$

The time derivative of (7.32) along (7.31) is given by

$$\dot{S}_{\tilde{q}v_v\zeta}(\tilde{q}, v_v, \zeta, q) = -\tilde{q}^\top \Pi^2 \tilde{q} - v_v^\top K v_v - (\zeta - \zeta_d)^\top K_\zeta^2 (\zeta - \zeta_d) < 0. \tag{7.34}$$

That is, (7.32) is a strict Lyapunov function for system (7.31). Furthermore, with the bounds in (7.33) it is easy to see that $\dot{S}_{\tilde{q}v_v\zeta}(\tilde{q}, v_v, \zeta, q) < -\beta S_{\tilde{q}v_v\zeta}(\tilde{q}, v_v, \zeta, q)$ and the function (7.32) is radially unbounded. This completes the proof. ■

7.4 Passivity-based synchronized tracking controls

In this section two alternative distributed trajectory tracking control laws are presented which can be seen as particular cases of the method described in Section 7.3 where virtual systems structure is exploited. The function ψ in (7.24) is changed by the combination of a diffusive linear feedback of position q and velocity v.

7.4.1 Slotine-Li synchronized tracking control

Recall that the networked EL dynamics (7.6) has the same structure as the individual agent dynamics as in (7.4). This motivates the introduction of a similar controller construction as the one in Theorem 7.3 for a single agent. However, in this case, the control objective is not only tracking to a reference signal $q_d(t) \in \mathcal{Q}$ but synchronized tracking to a common reference signal $q_d(t)$ for all the agents in the network. To this end, we propose the following sliding manifold (see Definition 4.12) for system (7.6) given by

$$\Omega = \{(q, v) : \dot{\tilde{q}} + (\Pi \otimes I_n)\tilde{q} + (B\Delta B^\top \otimes I_n)q = 0\}, \tag{7.35}$$

where $\tilde{q} = q - (\mathbb{1}_N \otimes q_d(t))$ with $\mathbb{1}_N \in \mathbb{R}^N$ the vector of all ones, $\Pi = \Pi^\top \in \mathbb{R}^{N\times N}$ and $\Delta = \Delta^\top \in \mathbb{R}^{M\times M}$ are positive definite diagonal matrices. Since $(B\Delta B^\top \otimes I_n)(\mathbb{1}_N \otimes q_d(t)) = 0_N$, the dynamics of (7.6) on the manifold Ω in (7.35) satisfies

$$\dot{\tilde{q}} = -([\Pi + B\Delta B^\top] \otimes I_n)\tilde{q}. \tag{7.36}$$

which is a linear in $\tilde{q}$ with diffusive coupling. If Π and Δ are chosen such that

$$P([\Pi + B\Delta B^\top] \otimes I_n) + ([\Pi + B\Delta B^\top] \otimes I_n)^\top P = -Q \tag{7.37}$$

where $P, Q \in \mathbb{R}^{Nn\times Nn}$ are symmetric and positive definite matrices, then $q(t) \to (\mathbb{1}_N \otimes q_d(t))$ exponentially as $t \to \infty$. Hence all the agents track $q_d(t)$ in a synchronized fashion. Thus, by defining

$$v_r := (\mathbb{1} \otimes \dot{q}_d(t)) - ([\Pi + B\Delta B^\top] \otimes I_n)\tilde{q}, \tag{7.38}$$

as an artificial velocity reference signal for the network dynamics (7.6), the sliding variable is given by $v_v = \dot{q} - v_r$.

Corollary 7.10. *Consider a strongly connected graph $\mathcal{G}$ with incidence matrix B and a reference trajectory $(\mathbb{1}_N \otimes q_d(t))$ for system* (7.6) *with $q_d(t) \in \mathcal{Q}$. Let v_v, v_r be the sliding variable and artificial velocity reference signal as defined before and the control law be given by*

$$\tau = M(q)\dot{v}_r + C(q,\dot{q})v_r + g(q) - Kv_v + u_v \tag{7.39}$$

where $K = K^\top \in \mathbb{R}^{Nn\times Nn}$ is a positive definite gain matrix. Then the closed-loop system of (7.6) *and* (7.39)

$$\begin{aligned} \dot{\tilde{q}} + ([\Pi + B\Delta B^\top] \otimes I_n)\tilde{q} &= v_v, \\ M(q)\dot{v}_v + C(q,v)v_v + Ks &= u_v, \end{aligned} \tag{7.40}$$

defines a strictly passive map $u \mapsto y = v_v$ with respect to the storage function

$$S_{v_v}(v_v, q) = \frac{1}{2} v_v^\top M(q) v_v, \tag{7.41}$$

parametrized by q. Moreover, we have that $v_v(t) \to 0$ and $\tilde{q}(t) \to 0$ as $t \to \infty$.

Corollary 7.10 can be seen as a particular case of the result in (Bai et al. 2011, Theorem 6.3). The only difference is that here the (artificial) reference velocity v_r defined above is considered (that is, v_{ri} is not common to all the agents), instead of a pure time-varying signal. The structure of v_r in this case implies that the external input contains also a diffusive velocity coupling, that is,

$$\overline{u}_v = -(B\Delta B^\top \otimes I_n)(q + v) + u_v. \tag{7.42}$$

Nevertheless, Corollary 7.10 can [3] be proved exactly in the same way as in (Bai et al. 2011, Theorem 6.3). The existence of Ω in (7.35) is guaranteed with v_r as defined above. This in turn implies that both $(\mathbb{1}_N \otimes q_d(t))$ and v_r are attractive trajectories for (7.40). Furthermore, the control scheme (7.39) can be split into the so-called *equivalent control*[4] with $\tau_{eq} = M(q)\dot{v}_r + C(q,\dot{q})v_r + g(q)$ and a feedback term $\tau_{fb} = -Kv_v$; the first ensures invariance once constrained to Ω and the later ensures that the off-manifold "distance" s converges to zero, i.e., attractivity. See also Corollary 4.14.

[3]Corollary 7.10 can also be proved similar to (Chung and Slotine 2009b, Theorem 5) by invoking the *contraction properties* of hierarchical combinations. This in turn implies *global exponential* convergence.

[4]The term equivalent control is adopted in the sliding mode control literature (Utkin 2013). This scheme can also be interpreted as an on-manifold control of the I&I framework in (Astolfi and Ortega 2003a).

Hence, the passivity preserving interconnection between the dynamics of v_v in (7.40) and the dynamics of $z = (B^\top \otimes I_n)q$ is kept, see (Arcak 2007) for further details.

7.4.2 Backstepping synchronized tracking control

In the previous distributed control approach, passivity of the closed-loop dynamics (7.40) as a whole is not used. This can be further exploited for performance improvement. To this end, notice that the position error dynamics is passive from the input $\bar{u}_q = v_v$ to the output $y_q = \tilde{q}$ with the storage function

$$S_{\tilde{q}}(\tilde{q}) = \frac{1}{2}\tilde{q}^\top([\Pi + B\Delta B^\top] \otimes I_n)\tilde{q}. \tag{7.43}$$

It is *hierarchically* interconnected to the passive dynamics (7.40) via $\bar{u}_q = v_v$. The above observations motivate the following proposition where we apply a backstepping redesign for the protocol in Corollary 7.10, which can be seen as the networked version of (7.17).

Proposition 7.11. *Consider a strongly connected graph $\mathcal{G}$ with incidence matrix B, and a reference trajectory* $(\mathbb{1}_N \otimes q_d(t))$ *for system* (7.6) *with $q_d(t) \in \mathcal{Q}$, $v_b = v - v_r$, and v_r as in* (7.38) *together with the control law given by*

$$\tau = M(q)\dot{v}_r + C(q, \dot{q})v_r + g(q) - Kv_v - ([\Pi + B\Delta B^\top] \otimes I_n)\tilde{q} + u_v \tag{7.44}$$

where $K = K^\top \in \mathbb{R}^{Nn \times Nn}$ is a positive definite gain matrix. Then the closed-loop system of (7.6) *with the control law* (7.44) *given by*

$$\begin{aligned} \dot{\tilde{q}} + ([\Pi + B\Delta B^\top] \otimes I_n)\tilde{q} &= v_v, \\ M(q)\dot{s}_v + C(q, \dot{q})v_v + Kv_v + ([\Pi + B\Delta B^\top] \otimes I_n)\tilde{q} &= u_v, \end{aligned} \tag{7.45}$$

is strictly passive from u to v_v with the storage function

$$S_{\tilde{q}v_v}(\tilde{q}, v_v, q) = S_{\tilde{q}}(\tilde{q}) + S_{v_v}(v_v, q) \tag{7.46}$$

parametrized by q. Moreover the origin of (7.45) *is uniformly globally exponentially stable with the rate $\beta = k_3/k_2$ where*

$$\begin{aligned} k_2 &= \max\{\lambda_{\max}([\Pi + B\Delta B^\top] \otimes I_n), \lambda_{\max}(M(q))\}, \\ k_3 &= \min\{\lambda_{\min}([\Pi + B\Delta B^\top]^2 \otimes I_n), \lambda_{\min}(K)\}. \end{aligned} \tag{7.47}$$

Proof: We will show that (7.46) is a *strict Lyapunov function* for the nonautonomous system (7.45) following (Khalil 1996, Theorem 4.10). First, we notice that (7.46) satisfies

$$k_1 \left\| \begin{bmatrix} \tilde{q} \\ v_v \end{bmatrix} \right\|^2 \leq S_{\tilde{q}v_v}(\tilde{q}, v_v, q) \leq k_2 \left\| \begin{bmatrix} \tilde{q} \\ v_v \end{bmatrix} \right\|^2 \tag{7.48}$$

where

$$k_1 = \min\{\lambda_{min}([\Pi + B\Delta B^\top] \otimes I_n), \lambda_{min}(M(q))\},$$

and k_2 is as in (7.47). The time derivative of (7.46) along the trajectories of (7.45) is

$$\dot{S}_{\tilde{q}v_v}(\tilde{q}, v_v, q) = - \begin{bmatrix} \tilde{q} \\ v_v \end{bmatrix}^\top \begin{bmatrix} ([\Pi + B\Delta B^\top]^2 \otimes I_n) & 0 \\ 0 & K \end{bmatrix} \begin{bmatrix} \tilde{q} \\ v_v \end{bmatrix} < 0, \tag{7.49}$$

uniformly in q. Hence, the storage function (7.46) qualifies as a strict Lyapunov function. Using the bounds of the Lyapunov function in (7.48), it is straightforward to see that $\dot{S}_{\tilde{q}v_v}(\tilde{q}, v_v, q) \leq -\beta S_{\tilde{q}v_v}(\tilde{q}, v_v, q)$, which completes the proof. ■

The extra term in the protocol (7.44) can be understood as a *feedforward action* to the closed-loop dynamics of the sliding variable s that preserves the passivity.

Remark 7.12. *Since system* (7.45) *is linear in the state* $(\tilde{q}, v_v)$, *invoking* contraction analysis *arguments, it can be shown that the matrix*

$$\begin{bmatrix} ([\Pi + B\Delta B^\top] \otimes I_n) & 0 \\ 0 & M(q) \end{bmatrix} \tag{7.50}$$

is a Riemannian metric *(see* (2.17)*). Thus, the gain matrices* Π, Δ *and* K *can be optimally tuned by taking* (7.46) *as a cost function subject to the network dynamics* (7.45).

7.5 Simulations

Consider the multi-robot network shown in Figure 7.5 with the following specifications:

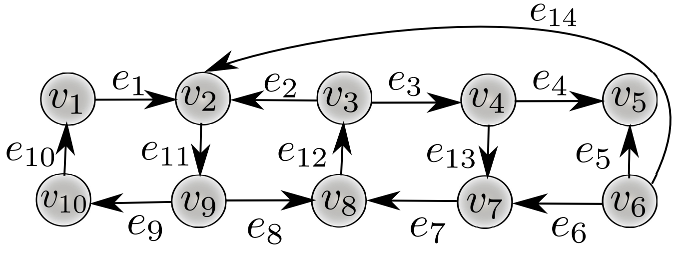

Figure 7.1: Network topology with $N = 10, M = 14, n = 2$.

Figure 7.2 and Figure 7.3 show the robots and springs position error, respectively.

Table 7.1: Simulation parameters

$q_d =$	$\sin(0.1t)$	$\dot{q}_d$	$0.1\cos(0.1t)$
$K_i =$	0.5	$K_\zeta =$	1000
$z_d =$	0	$\Pi_i =$	7.5

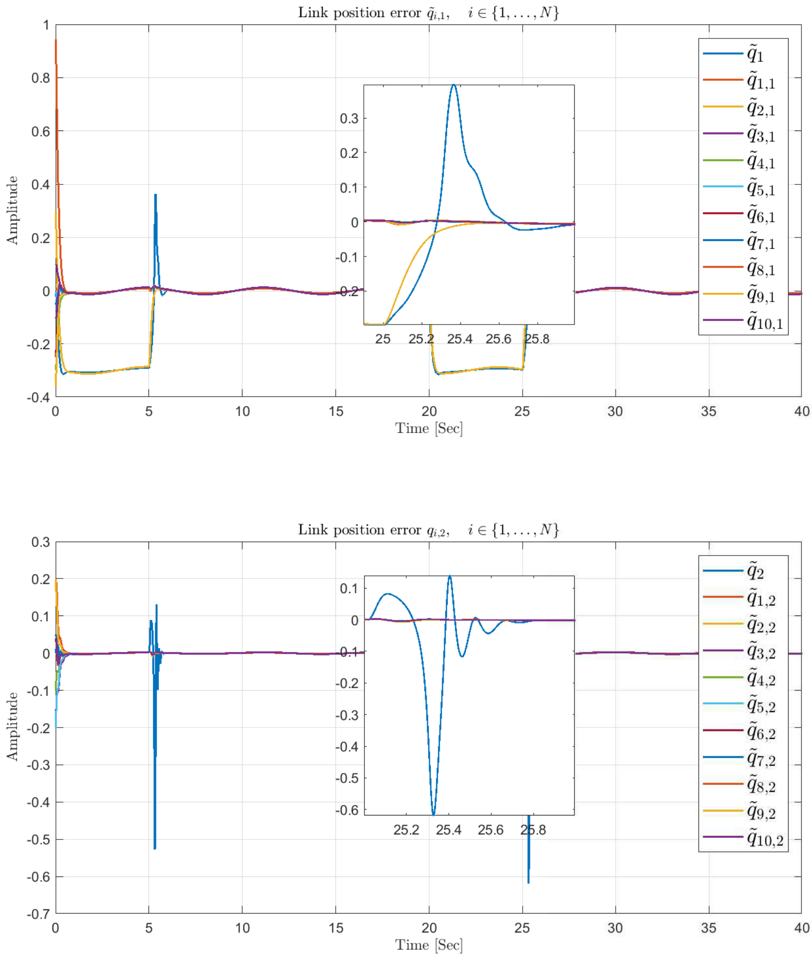

Figure 7.2: Links position error dynamics

Indeed, in Figure 7.2 it is shown the network steady-state behavior where agreement is reached. Furthermore, in order to show the robustness of the scheme, the agent attached to node v_2 is disturbed every 20 seconds with a pulse signal with amplitude 0.5; the network recovers the agreement after a smooth transient behavior.
Notice that every robot in the network is able to track the trajectory q_d independently

with the inner controller τ_i in (7.17). However, as shown in the plot of $\tilde{q}_1$ and $\tilde{q}_2$ in Figure 7.2, when the same disturbance is applied to the robot at node v_2 with $u_2 = 0$, the transient behavior gets worse (e.g. setting time and overshoot) than when $u_2 \neq 0$. That is, the action u_2 makes the robot to reject the disturbance in a smoothly manner. A similar scenario occurs under noise measurements.

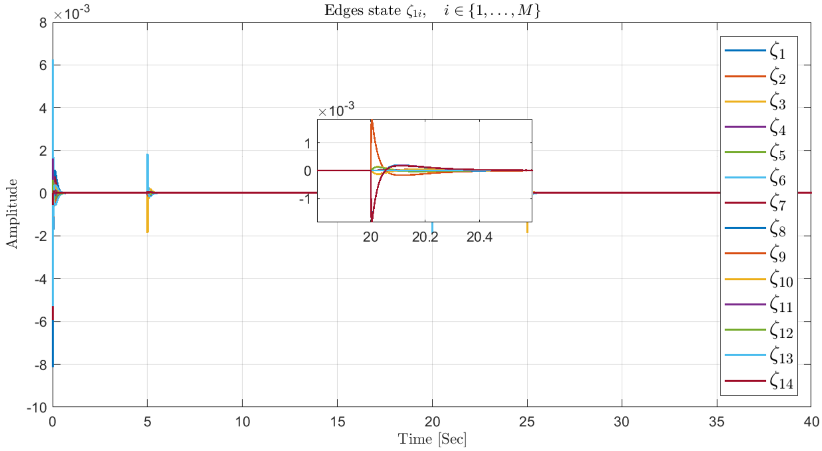

Figure 7.3: Artificial springs state ζ_k for all $k \in \{1, \ldots, M\}$.

Due to space limitation, in Figure 7.3 we only show the first state of the artificial springs. We observe that the behavior of these converge fast and it is robust. In particular, as expected, the springs attached to edges of node v_2, i.e., e_1, e_2 and e_{14} suffer most the disturbance. However, agreement is also recovered.

7.6 Conclusions

In this chapter the design procedure in (Arcak 2007) has been reformulated, where an artificial spring system is designed at each edge in the graph, instead of diffusing information through the relative positions and velocities, as commonly adopted. In the proposed approach, it is required that all the nodes' and edges' dynamics are strictly passive such that via a passivity preserving interconnection, the total storage function can be used as a strict Lyapunov function to show exponential converge to the desired trajectory. Moreover, thanks to the structure of the internal nodes feedback and the structure of the spring dynamics, exponential convergence is guaranteed. Numerical simulation confirms the theoretical developments. A possible disadvantage here is that we assume that the state of each agent is available, i.e., position and velocity. This can be solved by means of observers in the literature.

Chapter 8

Conclusions and future research

8.1 Conclusions

In this dissertation the problems of tracking control of mechanical systems in the pH framework and of the coordination control of networked EL systems have been solved using the contractivity and passivity properties of mechanical virtual systems.

For the first problem, a control design method called virtual contraction based control (v-CBC) has been proposed. To this end, a generalized definition of virtual systems was introduced; these systems can produce the same input-state-output behavior of a given original system. Then, by using the aforementioned definition, a class of energy-based virtual mechanical systems in the pH framework has been constructed. Remarkably, these systems preserve some of the original system's properties such as the passivity, that allows to have a clear physical interpretation of these systems.
With this and the notion of virtual contraction, the proposed method was stated with the following two main goals: *i*) designing a controller that makes the virtual mechanical system contractive in the closed-loop, and *ii*) impose a desired closed-loop steady state solution. Thus, the conclusion is that the steady-state solution of the contractive closed-loop virtual system and the original system's solution converge to each other. Therefore, this design method solved the trajectory tracking control problem.

The v-CBC method was then applied to solve the tracking control problem of fully-actuated mechanical systems in the port-Hamiltonian framework. This resulted in a large family of tracking controllers which in closed-loop with the virtual mechanical pH system exhibit different structural properties. For instance, under certain strict sufficient conditions the closed-loop preserves the structure of the open-loop virtual system; without any intermediate change of coordinates as it is usually done in the literature. Similarly, by relaxing these conditions, the closed-loop virtual system preserves only the structure of the variational open-loop virtual system; which means that the usual "spring/dampers" interpretation of the closed-loop system is only valid in a neighborhood of the origin (in error coordinates). Three novel different tracking controllers were developed for a two-degrees of freedom robot, each of these showing different closed-loop properties. The performance with this controllers was evaluated experimentally in a robot by Quanser, confirming the theoretical developments. However, a major disadvantage with respect to some schemes in the literature is that momentum (velocity) measurements are required.

The method was extended to a larger class of mechanical pH systems motivated by the tracking problem in flexible-joints robots (FJRs) and marine craft. This showed the applicability of the method to different kind of mechanical systems.
In the case of FJRs, the control design is a challenging problem because the system is underactuated. On the other hand, the marine craft is modeled as a rigid that evolves on moving frames; in particular on a body-fixed and on a inertial one. Since the Hamiltonian function is well defined only on the inertial frame and the attitude and velocities are measured in the body-fixed frame, the construction of the virtual system in the inertial frame and its control are interesting problems.
Similar to the inertial fully-actuated case, a family of v-CBC schemes that solve the standard trajectory tracking problem for these systems was developed. For the FJR the closed-loop performance was evaluated experimentally, also confirming the theoretical developments; and for the marine craft in numerical simulations.

The v-CBC design method developed here can be understood as a differential version of the backstepping technique (Khalil 1996) since the control scheme was obtained in a recursive manner via differential Lyapunov functions along the variational system rather than the system itself. Due to the structure of the artificial reference trajectory, the scheme can be also interpreted within the context of immersion and invariance technique (Astolfi and Ortega 2003b), where the closed-loop position virtual dynamics is the target system and the virtual momentum dynamics is the off-manifold behavior. Moreover, if a switching action is added as function of the off-manifold coordinate, then the scheme can be also interpreted within the context of sliding-modes control (Utkin 2013).

In second problem, the coordination control of networked mechanical systems, the passivity based approach of (Arcak 2007) is reformulated and generalized to agents with nonlinear dynamics in the EL framework. This was possible by using structure preserving controllers as preliminary feedback for each agent system such that each closed-loop agents has as target dynamics the developed virtual systems in the EL framework. With this target closed-loop dynamics the agents are allowed to be different. Moreover, after the preliminary feedback of each agent, the whole network dynamics has also the virtual EL system structure.

For the coordination protocol design, here it was required that all the nodes' and edges' dynamics are strictly passive such that via a passivity preserving interconnection, the total storage function can be used as a strict Lyapunov function.

8.2 Future research

Some of the open problems are described below:

- It remains open to explore the use of differential passivity is control design. For instance, to use the contraction preserving interconnection in (2.28) between the plant and the controller dynamics. This would lead to the differential counterpart of the control by interconnection technique in (Ortega et al. 2008).

- Due to the input-state-output relation between the original and the virtual control systems, it is of interest an interpretation of this relation in the behavioral approach of (Willems 2007). Similarly, investigate about the possible relation between the notion of a virtual control system can be related to the notion of an *abstraction system* in (Pappas and Simic 2002).

- Investigate further the geometric interpretation of the Poisson bracket in (3.51). Specifically, a different expression in local coordinates such that condition (3.52) holds uniformly in (q, p), and not only point wise.

- Explore the applicability of the timed-IDA-PBC technique developed in (Yaghmaei and Yazdanpanah 2017) in order to solve the step 2 in the v-CBC method for the virtual control system (4.1).

- Improve the experimental performance of the closed-loop system (4.17) related to noise measurements and parameters variations. For the first, differential passivity preserving velocity observers should be explored. For the second, an adaptive control scheme should be developed.

- Explore the possible benefits of nonlinear rotational springs in FJRs. The main consequence would be that the solvability of the necessary potential energy matching condition (5.8) may restrict the type of spring's dynamics, and the corresponding potential energy would be different from (5.3).

- Improve the closed-loop FJR experimental performance since in the setting developed here, it is necessary to have available the measurements of up to the third time derivative of the links position (i.e., velocity, acceleration and jerk), and measurements of the motors dynamics velocity. One possibility is to design differentially passive observers that use torque/force sensors measurements as input, similar to the work in (Albu-Schäffer et al. 2007).

- Reformulate the pH models and corresponding virtual systems of marine craft when the attitude is assumed to evolve in the special orthogonal group $SO(3)$ or the whole state (cartesian position and attitude) to evolve in the special Euclidean group $SE(3)$. Once in this setting, to solve the tracking problem.

- Extend the v-CBC method to other classes of mechanical systems. For instance, systems with nonholonomic restrictions, unilateral constrains, underactuated rigid bodies, etc. Similarly, apply the v-CBC method to solve the path following and maneuvering regulation problems in mechanical systems.
- Study the coordination control problem of heterogeneous networked mechanical systems. In particular, when the agents state spaces are of different dimension.
- Develop a network modeling framework to the analysis and control design of kinematic networks using passivity and virtual systems. This is motivated by the common practice in wheeled mobile robotics of using the kinematic model for control design purposes instead of the dynamic model. The main observation is that the kinematic system with its passive output can be interpreted as a spring-inverse damper system, where each agent (node) can be interpreted as a spring, its input as a inverse damper velocity, and the passive output a spring force.

Appendix A

Geometry tools for nonlinear systems

In this appendix we present a self-contained introduction to some differential geometry tools that are useful in this thesis. The information is mainly taken from (Marsden and Ratiu 2013, Chapter 4) and (Nijmeijer and van der Schaft 1990, Chapter 2), with complementary information from (Bullo and Lewis 2004, Chapter 3) and the books (do Carmo 1992) and (Del Castillo 2011).

A.1 Differentiable manifolds

Given a set $\mathcal{X}$, a *chart* on $\mathcal{X}$ is a pair $(\mathcal{W}, \varphi)$ where $\mathcal{W}$ is a subset of $\mathcal{X}$ and $\varphi : \mathcal{W} \to \varphi(\mathcal{W}) \subset \mathbb{R}^n$ a bijective map. We denote $\varphi(x)$ by $(x_1, \ldots, x_n)$ and call the x_i's the coordinates of the point $x \in \mathcal{W} \subset \mathcal{X}$. Two charts $(\mathcal{W}, \varphi)$ and $(\mathcal{W}', \varphi')$ such that $\mathcal{W} \cap \mathcal{W}' \neq \emptyset$ are called *compatible* if $\varphi(\mathcal{W} \cap \mathcal{W}')$ and $\varphi'(\mathcal{W}' \cap \mathcal{W})$ are open subsets of $\mathbb{R}^n$ and $\varphi' \circ \varphi^{-1}|_{\varphi(\mathcal{W} \cap \mathcal{W}')} : \varphi(\mathcal{W} \cap \mathcal{W}') \to \varphi'(\mathcal{W} \cap \mathcal{W}')$ and $\varphi \circ (\varphi')^{-1}|_{\varphi'(\mathcal{W} \cap \mathcal{W}')} : \varphi'(\mathcal{W} \cap \mathcal{W}') \to \varphi(\mathcal{W} \cap \mathcal{W}')$ are C^∞ maps.

We say that $\mathcal{X}$ is a *differentiable n-manifold* if the following hold: $\mathcal{X}$ has an *atlas*, that is, $\mathcal{X}$ can be written as a union of compatible charts. A coordinate *neighborhood* of a point $x \in \mathcal{X}$ is an open set defined by inverse image of a Euclidean space neighborhood of the point $\varphi(x)$ under the chart map $\varphi : \mathcal{W} \to \mathbb{R}^n$.

A *curve* on $\mathcal{X}$ through x is a map $\gamma : I \to \mathcal{X}$, where $I \subseteq \mathbb{R}$ contains 0 in its interior and $\gamma(0) = x$. Two curves γ_1 and γ_2 are *equivalent* at x (denoted by $\gamma_1 \sim_x \gamma_2$) if in some chart $(\mathcal{W}, \varphi)$ it holds that $\gamma_1(0) = \gamma_2(0) = x$ and[1] $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$.

A.2 Tangent bundle and vector fields

Tangent bundle

A *tangent vector* v to a manifold $\mathcal{X}$ at the point $x \in \mathcal{X}$ is an equivalence class of curves at x under $\sim_x$. The collection of all tangent vectors at x is a vector space $T_x\mathcal{X}$, called the

[1] The prime denotes differentiation of curves in $\mathbb{R}^n$.

tangent space at x. Given a curve $\gamma(t)$, denote by $\gamma'(s)$ the tangent vector at $\gamma(s)$ defined by the equivalence class of $t \mapsto \gamma(s+t)$ at $t = 0$. The tangent vector $v \in T_x\mathcal{X}$ can be identified with the tangent vector to a curve on $\mathcal{X}$. Thus, in a chart $(\mathcal{W}, \varphi)$ of $\mathcal{X}$ with local coordinates $(x_1, \ldots, x_n)$, let γ be a curve that is a representative of the equivalence class v. The *components* of the tangent vector $v \in T_x\mathcal{X}$ are numbers $v_1, \ldots, v_n$ given by $v_i = \frac{d}{dt}(\varphi \circ \gamma)_i\big|_{t=0}$, where $(\varphi \circ \gamma)_i$ denotes the i-th component of the curve γ in the chart, for all $i \in \{1, \ldots, n\}$.

The disjoint union of all tangent spaces $T\mathcal{X} = \bigcup_{x\in\mathcal{X}} T_x\mathcal{X}$ is called the *tangent bundle*. A point of $T\mathcal{X}$ is a vector v tangent to $\mathcal{X}$ at some point $x \in \mathcal{X}$. To define the differentiable structure on $T\mathcal{X}$, we need to construct local coordinates on $T\mathcal{X}$. To this end consider the local coordinates $(x_1, \ldots, x_n)$ on $\mathcal{X}$ at a point x and the corresponding components of a tangent vector v $(v_1, \ldots, v_n)$ in this chart. Then $((x_1, \ldots, x_n), (v_1, \ldots, v_n))$ give a local coordinate system on $T\mathcal{X}$. This shows that the dimension[2] of $T\mathcal{X}$ is $2n$. The *natural projection* $\pi_{\mathcal{X}} : T\mathcal{X} \to \mathcal{X}$ takes $v \in T_x\mathcal{X}$ to the point $x \in \mathcal{X}$ at which v is attached. The inverse image $\pi_{\mathcal{X}}^{-1}(x)$ of a point $x \in \mathcal{X}$ under the natural projection $\pi_{\mathcal{X}}$ is the tangent space $T_x\mathcal{X}$, called the *fiber* of the tangent bundle $T\mathcal{X}$ at the point $x \in \mathcal{X}$.

Tangent maps

Let $F : \mathcal{X} \to \mathcal{Y}$ be a smooth map from a manifold $\mathcal{X}$ to a manifold $\mathcal{Y}$. We say that F is differentiable if in charts on $\mathcal{X}$ and $\mathcal{Y}$ the map is represented by a differentiable map. The *derivative or tangent map* of a differentiable map $F : \mathcal{X} \to \mathcal{Y}$ at a point $x \in \mathcal{X}$ is defined to be the linear map[3] $T_xF : T_x\mathcal{X} \to T_{F(x)}\mathcal{Y}$ constructed in the following manner: For $v \in T_x\mathcal{X}$, consider the curve $\gamma : (-\epsilon, \epsilon) \to \mathcal{X}$ with $\gamma(0) = x$ and associated velocity vector $\gamma'(0) = v$. Then, $T_xF(v)$ is the velocity vector at $t = 0$ of the curve $F \circ \gamma : (-\epsilon, \epsilon) \to \mathcal{Y}$, that is, $T_xF(v) = \frac{d}{dt}F(\gamma(t))|_{t=0}$.

Let $x = (x_1, \ldots, x_n)$ be local coordinates for $\mathcal{X}$ and $y = (y_1, \ldots, y_m)$ for $\mathcal{Y}$, then in natural coordinates $(x, v) = (x_1, \ldots, x_n, v_1, \ldots, v_n)$ for $T\mathcal{X}$ respectively

$$(y, w) = (y_1, \ldots, y_n, w_1, \ldots, w_m)$$

for $T\mathcal{Y}$, a local representative of the tangent map TF at $x \in \mathcal{X}$ is given in as $TF(x, v) = (F(x), T_xF(x)(v))$, where T_xF is the Jacobian.

If $F : \mathcal{X} \to \mathcal{Y}$ and $G : \mathcal{Y} \to \mathcal{Z}$ are differentiable maps, then $G \circ F : \mathcal{X} \to \mathcal{Z}$ is differentiable, and the *chain rule* holds: $T(G \circ F) = TG \circ TF$.

[2]One must notice that this does not mean that $T\mathcal{X}$ is a product manifold.

[3]Other common notations in the literature for the tangent map TF include F_*, dF, DF.

For the curve $\gamma : I \to \mathcal{X}$, we see that the tangent map $T_s\gamma : T_s\mathbb{R} \to T_x\mathcal{X}$ defines the tangent vector γ' at $t = s$ as the vector $\gamma'(s) = \frac{d\gamma}{dt}|_{t=s} = T_s\gamma \cdot 1 = D\gamma(s)(1) = \dot{\gamma}(s)$.

A differentiable map $F : \mathcal{X} \to \mathcal{Y}$ is called a *diffeomorphism* if it is bijective and its inverse is also differentiable. If $T_xF : T_x\mathcal{X} \to T_{F(x)}\mathcal{Y}$ is an invertible map, the *inverse function theorem* states that F is a *local diffeomorphism* around $x \in \mathcal{X}$.

Vector fields and dynamical systems

A *vector field* X on a manifold $\mathcal{X}$ is a linear C^∞ map $X : \mathcal{X} \to T\mathcal{X}$ that assigns a vector $X(x)$ at the point $x \in \mathcal{X}$; that is, $\pi_{\mathcal{X}} \circ X$ =identity. The linear vector space of vector fields on $\mathcal{X}$ is denoted by $\mathfrak{X}^\infty(\mathcal{X})$. An *integral curve* of X with initial condition x at $t = 0$ is a smooth map $\gamma : (a, b) \to \mathcal{X}$ such that $0 \in (a, b)$, $\gamma(0) = x$, and $\gamma'(t) = X(\gamma(t))$ for all $t \in (a, b)$. The *flow* of X is the collection of maps $\vartheta_t : \mathcal{X} \to \mathcal{X}$ such that $\vartheta_t(x)$ is the integral curve of X with initial condition x. By existence and uniqueness theorems, smoothness of ϑ is guaranteed in both x and t. Uniqueness implies the *flow property*[4] $\vartheta_{t+s} = \vartheta_t \circ \vartheta_s$ a long with the initial condition ϑ_0 =identity.

A *time-dependent vector field* is a map $X : \mathcal{X} \times \mathbb{R} \to T\mathcal{X}$ such that $X(x, t) \in T_x\mathcal{X}$ for each $x \in \mathcal{X}$ and $t \in \mathbb{R}$. The real vector space of vector fields on $\mathcal{X} \times \mathbb{R}$ is denoted by $\mathfrak{X}^\infty(\mathcal{X} \times \mathbb{R})$. An integral curve of X is a curve $\gamma(t)$ in $\mathcal{X}$ such that $\gamma'(t) = X(\gamma(t), t)$. The flow is the collection of maps $\vartheta_{t,s} : \mathcal{X} \to \mathcal{X}$ such that $t \mapsto \vartheta_{t,s}(x)$ is the integral curve $\gamma(t)$ with initial condition $\gamma(s) = x$ at $t = s$. The existence and uniqueness theorems imply that the *time-dependent flow property* $\vartheta_{t,s} \circ \vartheta_{s,r} = \vartheta_{t,r}$. If X is time independent, the two notions of flows are related through $\vartheta_{t,s} = \vartheta_{t-s}$.

In the chart $(\mathcal{W}, \varphi)$ and the induced natural local chart $(\pi^{-1}(\mathcal{W}), T\varphi)$ with coordinates $(x_!, \ldots, x_n, v_1, \ldots, v_n)$, the local representative of a vector field $X : \mathcal{X} \to T\mathcal{X}$ is the map $X : T\varphi \circ X \circ \varphi^{-1} : \varphi(\mathcal{W}) \subset \mathbb{R}^n \to T_\varphi(\mathcal{W}) \cong \varphi(\mathcal{W}) \times \mathbb{R}^n$ written as $\hat{X}(x_1, \ldots, x_n) = (x_1, \ldots, x_n, \hat{X}_1(x), \ldots, \hat{X}_n(x))$, for some $\hat{X}_i(x)$ with $i \in \{1, \ldots, n\}$. In fact, these functions $\hat{X}_i(x)$ are given by the formula (with a slightly abuse of notation)

$$X(x) = \sum_{i=1}^{n} \hat{X}_i(\varphi(x)) \frac{\partial}{\partial x_i}|_x,$$

where $\frac{\partial}{\partial x_i}|_x := T_x\varphi^{-1}(e_{i\varphi(x)})$ for $i \in \{1, \ldots, n\}$, and $\{e_{1\varphi(x)}, \ldots, e_{n\varphi(x)}\}$ is the basis of $T_{\varphi(x)}\mathbb{R}^n$. For short $\frac{\partial}{\partial x_i}|_x = \frac{\partial}{\partial x_i}$.

It is customary to omit all the carets and usually write X in local coordinates as $X(x) = \sum_{i=1}^{n} X_i(q) \frac{\partial}{\partial x_i}$ or as the vector $X(x) = [X_1(x), \ldots, X_n(x)]^\top$, where X_i are the func-

[4] If $\mathcal{X} = \mathcal{V}$ is a *linear* space, $X(x) = Ax$ for a linear operator A, then $\vartheta_t(x) = e^{tA}x$. In this case the flow property amounts to $e^{tA}\left(e^{sA}\right) = e^{(t+s)A}$.

tions $\hat{X}_i$ above. Hence to a vector field X expressed in local coordinates $x = (x_1, \dots, x_n)$ with corresponding integral curve equation $\gamma'(t) = X(\gamma(t))$ we associate in a one-to-one way the differential equation $\dot{x} = X(x)$; we refer to this differential equation as a *dynamical system*. Similar procedure holds for time-dependent vector fields and corresponding *time-dependent dynamical systems*.

Dynamical control systems

Let $\mathcal{X}$ and $\mathcal{U}$ be two manifolds and define the smooth map $F : \mathcal{X} \times \mathcal{U} \to T\mathcal{X}$ such that $\pi_{\mathcal{X}} \circ F$ is equal to the natural projection of $\mathcal{X} \times \mathcal{U}$ onto $\mathcal{X}$. With the same abuse of notation as above, the representative of F in local coordinates x for $\mathcal{X}$, corresponding natural coordinates (x, v) for $T\mathcal{X}$, and local coordinates $u = (u_1, \dots, u_m)$ for $\mathcal{U}$ as $F(x, u) = (x, f(x, u))$. To the corresponding integral curve equation $\gamma'(t) = X(\gamma(t), u(t))$ we associate the local coordinate differential equation $\dot{x} = X(x, u)$.

We refer to this differential equation as a dynamical system with input u, or equivalently as a *dynamical control system* with state space $\mathcal{X}$ and input space $\mathcal{U}$. In case the (system) map $F : \mathcal{X} \times \mathcal{U} \to T\mathcal{X}$ is affine in the u-variables we write $F(x, u) = F(x) + \sum_{i=1}^{m} G_j(x)u_i$, with the addition and multiplication defined on the linear space $T_x\mathcal{X}$ and for some functions $F, G, \dots, G_m : \mathcal{X} \to T\mathcal{X}$ satisfying $\pi_{\mathcal{X}} \circ F = \pi_{\mathcal{X}} \circ G_j =$identity on $\mathcal{X}$, which hence are *vector fields* on $\mathcal{X}$.

Lie derivative and Lie brackets

Let $C^\infty(\mathcal{X})$ denote the set of smooth real functions $f : \mathcal{X} \to \mathbb{R}$. A function $f \in C^\infty(\mathcal{X})$ can be differentiated at any point $x \in \mathcal{X}$ to obtain the tangent map $T_x f : T_x\mathcal{X} \to T_{f(x)}\mathbb{R}$. Since $T_{f(x)}\mathbb{R} \simeq \mathbb{R}$, we get a linear map $T_x f : T_x\mathcal{X} \to \mathbb{R}$. Thus, the local representative of Tf in natural coordinates (x, v) of $T\mathcal{X}$ is $T_x f(v) = (f(x), T_x f \cdot v)$. For the coming developments, denote the tangent map of $f \in C^\infty(\mathcal{X})$ as $T_x f = df(x)$.

For a tangent vector $v \in T_x\mathcal{X}$, we define the *directional derivative* of f in the direction v as $df(x) \cdot v$. In coordinates, this is given by $df(x) \cdot v = \sum_{i=1}^{n} \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(x) v_i$, for a chart $(\mathcal{W}, \varphi)$ at $x \in \mathcal{X}$. By definition a vector field $X \in \mathfrak{X}^\infty(\mathcal{X})$ defines in any $x \in \mathcal{X}$ a tangent vector $v = X(x)$ yielding the directional derivative $df(x) \cdot X(x)$ for a real function $f \in C^\infty(\mathcal{X})$. From above observations, by varying x the vector field X acts as the operator $X(x) : C^\infty(\mathcal{X}) \to \mathbb{R}$ defined by $X[f](x) = \lim_{h \to 0} \frac{f(\vartheta_t(x)) - f(x)}{h}$. This is similar for time-varying vector fields.

This function $X[f](x)$ is called the *total derivative*, or *Lie derivative* of f along X, also denoted as $L_X f$. If X is expressed in coordinates $x = (x_1, \dots, x_n)$ as the vector

$[X_1(x), \ldots, X_n(x)]^\top$ then $L_X f(x) = X[f](x) = df(x) \cdot X(x) = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) X_i(x)$.

For any $X, Y \in \mathfrak{X}^\infty(\mathcal{X})$, we define a new vector field, denoted as $[X, Y]$ and called the *Lie bracket* of X and Y, by setting $[X, Y][f](x) = X[Y[f]](x) - Y[X[f]](x)$. If X and Y are given in coordinates $(x_1, \ldots, x_n)$ as the column vectors $X(x) = [X_1(x), \ldots, X_n(x)\top]^\top$ and $Y(x) = [Y_1(x), \ldots, Y_n(x)\top]^\top$, then $[X, Y](x) = (\frac{\partial Y}{\partial x} X - \frac{\partial X}{\partial x} Y)(x)$. The Lie bracket can be interpreted as the "derivative" of Y along X, and it is also denoted as $L_X Y$, the Lie derivative of Y along X.

A.3 Cotangent bundle and differential forms

Cotangent bunbdle

The dual space of the tangent space $T_x\mathcal{X}$ is denoted by $T_x^*\mathcal{X}$, called the *cotangent space* of $\mathcal{X}$. Elements in $T_x^*\mathcal{X}$ are called *cotangent vectors* or *covectors*. In local coordinates $(x_1, \ldots, x_n)$ of $\mathcal{X}$, let $\{\frac{\partial}{\partial x_1}|_x, \ldots, \frac{\partial}{\partial x_n}|_x\}$ be a basis for $T_x\mathcal{X}$, then we denote the dual basis of $T_x^*\mathcal{X}$ by $\{dx_1|_x, \ldots, dx_n|_x\}$ or for short $\{dx_1, \ldots, dx_n\}$. By definition $dx_i(\frac{\partial}{\partial x_j}) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ or $\delta_{ij} = 0$ otherwise, for $i, j \in \{1, \ldots, n\}$. Thus any element $\alpha \in T_x^*\mathcal{X}$ can be written as $\alpha = \sum_{i=1}^n \alpha_i dx_i$ for some coefficients α_i and is also denoted as a row-vector $[\alpha_1, \ldots, \alpha_n]$.

The cotangent bundle $T^*\mathcal{X}$ of a manifold $\mathcal{X}$ is defined as the union of all cotangent spaces $\bigcup_x \in T\mathcal{X}_x^*\mathcal{X}$. There is the natural projection $\pi_\mathcal{X}^* : T^*\mathcal{X} \to \mathcal{X}$ taking the cotangent vector $\alpha \in T_x^*\mathcal{X} \subset T^*\mathcal{X}$ to $x \in \mathcal{X}$. As in the case of a tangent bundle, the cotangent bundle possesses a manifold structure: Given a chart $(\mathcal{W}, \varphi)$ on $\mathcal{X}$ at x, we obtain natural coordinates for $T^*\mathcal{X}$ by the local chart $((\pi_\mathcal{X}^*)^{-1}(\mathcal{W}), d\varphi)$ with coordinates $(x_1, \ldots, x_n, \alpha_1, \ldots, \alpha_n)$.

Differential one-forms

Define the *differential one-form* α, the dual object to a vector field on manifold $\mathcal{X}$, as the smooth map $\alpha : \mathcal{X} \to T^*\mathcal{X}$, satisfying $\pi_\mathcal{X}^* \circ \alpha =$ identity on $\mathcal{X}$. Hence a one form $\alpha(x)$ is a map which assigns to each $x \in \mathcal{X}$ a cotangent vector $\alpha(x) \in T_x^*\mathcal{X}$.

Let $(\mathcal{W}, \varphi)$ be a chart for $\mathcal{X}$ at x, and the corresponding basis $(dx_1, \ldots, dx_n)$ for $T_x^*\mathcal{X}$, then $\hat{\alpha}_i := \alpha_i \circ \varphi^{-1} : \varphi(\mathcal{W}) \to \mathbb{R}$ is the local representative of α_i, for certain $\alpha_i : \mathcal{W} \to \mathbb{R}$, for all $i \in \{1, \ldots, n\}$, also omitting carets, we write α is local coordinates as the row vector $\alpha(x) = [\alpha_1(x), \ldots, \alpha_n(x)]$ or $\alpha(x) = \sum_{i=1}^n \alpha_i(x) dx_i$.

Since one-forms α are dual objects of vector fields X, they act in a natural way upon vector fields as $\alpha[X](x) = \alpha(x)[X(x)] := \langle \alpha(x), X(x) \rangle \in \mathbb{R}$; that is $\alpha[X]$ is a smooth function on $\mathcal{X}$. We call $\langle \cdot, \cdot \rangle : T_x^*\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R}$ the *canonical pairing*.

Similarly, a differential *two-form* Ω on $\mathcal{X}$ is a function $\Omega(x) : T_x\mathcal{X} \times T_x\mathcal{X} \to \mathbb{R}$ that assigns to each point $x \in \mathcal{X}$ a *skew-symmetric* bilinear form on the $T_x\mathcal{X}$ to $\mathcal{X}$ at x. At each $x \in \mathcal{X}$,the two-from is $\Omega_x(v, w) = \sum_{i,j=1}^{n} \Omega_{ij}(x)v_i\omega_j$ where $\Omega_{ij}(x) = \Omega_x(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$.

A function $f \in C^\infty(\mathcal{X})$ defines a one-form in $T_x^*\mathcal{X}$, denoted as $df(x)$ by the total derivative formula $df(x)[X(x)] = X[f](x)$, for $X(x) \in T_x\mathcal{X}$. We say that $df(x)$ is the *differential* of f at $x \in \mathcal{X}$. One can check that $df(x)$ satisfies the equality $df[X] = X[f] = L_X f$. In coordinates, if we interpret dx_i as the differential of the coordinate function x_i, then the differential $df(x)$ is given in coordinates as $df(x) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x)dx_i$. Notice that not every one-form can be written as df for a given function $f \in C^\infty(\mathcal{X})$. If such f exists we say that the one-form is *exact*. If the two-form $d\alpha$ is zero, then α is called *closed*. This leads to the *Poincaré lemma*: a closed differential form is locally exact.

Appendix B

Energy-based modeling of mechanical systems

In this appendix a self-contained survey on the Euler-Lagrange and Hamiltonian frameworks for mechanical systems energy-based modeling is presented. For sake of physical intuition, deductions are performed in local coordinates. However, some properties of interest are further interpreted from a coordinate-free point of view with the language and tools of Appendix A.
The information is taken from (van der Schaft 2017, Chapter 4), (van der Schaft and Maschke 1995b), and (Nijmeijer and van der Schaft 1990, Chapter 12) and complemented with (Ortega et al. 2013, Chapter 2), (Marsden and Ratiu 2013, Chapters 2, 4 and 7) and (Arimoto and Miyazaki 1984, Bullo and Lewis 2004, Chang 2002).

B.1 Mechanical Euler-Lagrange control systems

Let Q be the configuration manifold of a mechanical system of n degrees of freedom (dof) with a local coordinate chart $(\mathcal{W}, \varphi)$ at $q \in Q$. Without carets take $q = (q_1, \ldots, q_n)$ as the system's position. Consider function $L : TQ \to \mathbb{R}$ called the *Lagrangian* which in natural coordinates (q, v) of TQ is given by $L(q_1, \ldots, q_n, v_1, \ldots v_n) = L(q, v)$. Let $\Gamma(q_a, q_b) = \{\gamma : C^2[0,1] \to Q | \gamma(0) = q_a, \gamma(1) = q_b\}$ be the collection of twice continuously differentiable curves on $[0, 1]$ connecting $q_a \in Q$ and $q_b \in Q$ with local representative given by the vector mapping $t \mapsto q(t) = [q_1(t), \ldots, q_n(t)]^\top$.

The equations of motion for mechanical control systems in the EL framework are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}(q, \dot{q})\right) - \frac{\partial L}{\partial q}(q, \dot{q}) = B(q)\tau, \tag{B.1}$$

where $(q(t), \dot{q}(t))$ are the local representatives of $\gamma \in \Gamma(q_1, q_b)$ and its time derivative $\gamma'(t)$, respectively. The covector $B(q)\tau \in T^*Q$, with inputs $\tau \in \mathcal{U}$, represents the vector of external forces. Matrix $B(q)$ indicates how the action of the inputs τ influences the system. If rank $B(q) = m < n$ then we say that system (B.1) is *underactuated*.

For *simple* mechanical systems the Lagrangian function is the difference between the kinetic (co)-energy and the potential energy; in coordinates given as

$$L(q,\dot q) = \frac{1}{2}\dot q^\top M(q)\dot q - P(q) = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n M_{ij}(q)\dot q_i \dot q_j - P(q), \tag{B.2}$$

where $M(q)$ is the $n \times n$, symmetric and positive definite inertia matrix, with $M_{ij}(q)$ its (i,j)-th entry. Computing

$$\frac{\partial L}{\partial \dot q_k}(q,\dot q) = \sum_{j=1}^n M_{kj}(q)\dot q_j,$$

and

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}(q,\dot q)\right) = \sum_{j=1}^n M_{kj}(q)\ddot q_j + \sum_{j=1}^n \sum_{i=1}^n \frac{\partial M_{kj}}{\partial q_i}(q)\dot q_i \dot q_j,$$

as well as

$$\frac{\partial L}{\partial q_k}(q,\dot q) = \frac{\partial}{\partial q_k}\left(\frac{1}{2}\dot q^\top M(q)\dot q\right) - \frac{\partial P}{\partial q_k}(q) = \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial M_{ij}}{\partial q_k}(q)\dot q_i \dot q_j - \frac{\partial P}{\partial q_k}(q),$$

the EL equations (B.1) for the Lagrangian function (B.2) take the form

$$\sum_{j=1}^n M_{kj}(q)\ddot q_j + \sum_{i,j=1}^n h_{kji}(q)\dot q_i \dot q_j + g_k(q) = \tau_k, \qquad k \in \{1,\ldots,n\}. \tag{B.3}$$

where

$$h_{kji}(q) := \frac{\partial M_{kj}}{\partial q_i}(q) - \frac{1}{2}\frac{\partial M_{ij}}{\partial q_k}(q), \quad \text{and} \quad g_k(q) := \frac{\partial P}{\partial q_k}(q). \tag{B.4}$$

The EL equations in (B.3) can be expressed in compact from as

$$M(q)\ddot q + C(q,\dot q)\dot q + g(q) = B(q)\tau, \tag{B.5}$$

with $g(q) = [g_1(q),\ldots,g_n(q)]^\top$ and $C(q,\dot q)$ any matrix satisfying

$$\sum_{j=1}^n C_{kj}(q,\dot q)\dot q_j = \sum_{i,j=1}^n h_{kji}(q)\dot q_i \dot q_j \iff C(q,\dot q)\dot q = \dot M(q)\dot q - \frac{\partial}{\partial q}\left(\frac{1}{2}\dot q^\top M(q)\dot q\right). \tag{B.6}$$

The forces $C(q,\dot q)\dot q$ correspond to the centrifugal ($i = j$) and Coriolis ($i \neq j$) effects, respectively. Notice that (B.5) is a set of second-order differential equations.

For simple mechanical systems the $n \times n$ matrix with (i,j)-th element $\frac{\partial L}{\partial \dot q_i \dot q_j}(q,\dot q)$ is equal to $M(q)$ and thus nonsingular. Hence the Euler-Lagrangian equations define the affine system with state space representation of (B.5)

$$\frac{d}{dt}\begin{bmatrix} q \\ \dot{q} \end{bmatrix} = \begin{bmatrix} \dot{q} \\ -M^{-1}(q)\left(C(q,\dot{q})\dot{q} + g(q)\right) \end{bmatrix} + \begin{bmatrix} 0_n \\ M^{-1}(q)B(q) \end{bmatrix}\tau. \tag{B.7}$$

B.1.1 Euler-Lagrange equations and Riemannian geometry

Since the inertia matrix $M(q)$ defines a *Riemannian metric* on Q given by

$$M\langle v, w\rangle := v^\top M(q)w, \quad \text{for} \quad u, w \in T_qQ, \tag{B.8}$$

a geometric interpretation of the EL equations of motion (B.5) can be given within the context of *Riemannian manifolds* (van der Schaft 2017, Section 4.6).

To this end, lets us introduce some concepts. For any pair of vector fields $X, Y \in \mathfrak{X}^\infty(TQ)$ and any real function $f \in C^\infty(Q)$, an *affine connection* ∇ is a map $(X, Y) \mapsto \nabla_X Y \in \mathfrak{X}^\infty(TQ)$ such that

$$\begin{aligned}
&(a)\nabla_X Y \quad \text{is bilinear in } X \text{ and } Y,\\
&(b)\nabla_{fX} Y = f\nabla_X Y,\\
&(c)\nabla_X fY = f\nabla_X Y + (L_X f)Y.
\end{aligned}$$

The vector $\nabla_X Y$ is called the *covariant derivative* of Y with respect to X. Property (b) implies that $\nabla_X Y$ at $q \in Q$ depends on X only through its value $X(q)$. The *torsion* of an affine connection ∇ is defined as

$$T(X, Y) := \nabla_X Y - \nabla_Y X - [X, Y]. \tag{B.9}$$

If $T(X, Y) = 0$, we say the connection is *torsion-free*. An affine connection ∇ on Q is said *metric* or *compatible* with the Riemannian metric in (B.8) if

$$L_X\left(M\langle Y, Z\rangle\right) = M\langle \nabla_X Y, Z\rangle + M\langle X, \nabla_X Z\rangle \tag{B.10}$$

for all vector fields $X, Y, Z \in \mathfrak{X}^\infty(Q)$. In a chart (Q, φ) at q and corresponding basis $\{\frac{\partial}{\partial q_1}, \ldots, \frac{\partial}{\partial q_n}\}$ of T_qQ, the ℓ-th component of $\nabla_{\frac{\partial}{\partial q_i}} \frac{\partial}{\partial q_j} \in \mathfrak{X}^\infty(\mathcal{W})$ is locally written as

$$\left(\nabla_{\frac{\partial}{\partial q_i}} \frac{\partial}{\partial q_j}\right)_\ell = \sum_{i,j=1}^{n} \Gamma_{ij}^\ell(q) \frac{\partial}{\partial x_\ell}, \quad \text{for all} \quad \ell \in \{1, \ldots, n\}, \tag{B.11}$$

where the n^3 smooth functions $\Gamma_{ij}^\ell(q)$ are uniquely defined. With these functions, called the *Christoffel symbols of second kind*, the ℓ-th component of $\nabla_X Y$ is

$$(\nabla_X Y)_\ell = \sum_{j=1}^{n} \frac{\partial Y_\ell}{\partial q_j} X_j + \sum_{i,j=1}^{n} \Gamma_{ij}^{\ell} X_i Y_j, \quad \text{for all} \quad \ell \in \{1, \ldots, n\}. \tag{B.12}$$

This in turn implies that the ℓ-th component of the covariant derivative of $Y \in \mathfrak{X}^\infty(Q)$ along a curve $\gamma \in \Gamma(q_a, q_b)$ is given by

$$(\nabla_{\gamma'(t)} Y(\gamma(t)))_\ell = \dot{Y}_\ell(\gamma(t)) + \sum_{i,j=1}^{n} \Gamma_{ij}^{\ell}(\gamma(t))\gamma_i'(t) Y_j(\gamma(t)), \quad \ell \in \{1, \ldots, n\}. \tag{B.13}$$

A curve γ is called a *geodesic* of the affine connection ∇ if $\nabla_{\gamma'(t)}\gamma'(t) = 0_n$ holds. In coordinates, the geodesic γ is the solution of the set of second-order equations

$$\ddot{q}_\ell(t) + \sum_{i,j=1}^{n} \Gamma_{ij}^{\ell}(q(t))\dot{q}_i(t)\dot{q}_j(t) = 0_n, \quad \ell \in \{1, \ldots, n\}, \tag{B.14}$$

where $t \mapsto [q_1(t), \ldots, q_n(t)]^\top$ is a local representative for γ.

The Riemannian metric (B.8) defines a unique affine connection $\overset{M}{\nabla}$ on Q, called the *Levi-Civita connection*, which is torsion-free and compatible. In this case, due to the *Koszul formula*[1] (do Carmo 1992), the Christoffel symbols Γ_{ij}^{ℓ} (B.11) are given by

$$\Gamma_{ij}^{\ell}(q) := \sum_{k=1}^{n} M_{\ell k}^{-1}(q) c_{ijk}(q), \tag{B.15}$$

with $M_{\ell k}^{-1}(q)$ the (ℓ, k)-th element of matrix $M^{-1}(q)$, and the functions $c_{ijk}(q)$ are the so-called Christoffel symbols of *first kind* defined as

$$c_{ijk}(q) := \frac{1}{2}\left[\frac{\partial M_{kj}}{\partial q_i}(q) + \frac{\partial M_{ki}}{\partial q_j}(q) - \frac{\partial M_{ij}}{\partial q_k}(q)\right]. \tag{B.16}$$

It can be shown that the geodesic curve of $\overset{M}{\nabla}$ is precisely the solution to the EL equations (B.1) on Q in the absence of external forces ($\tau = 0$) (Bullo and Lewis 2004, Proposition 4.43). Indeed, by interchanging the indexes of the double sum in (B.3) we have that

$$\sum_{i,j=1}^{m} \frac{\partial M_{kj}}{\partial q_i}(q) = \sum_{j,i=1}^{m} \frac{\partial M_{ki}}{\partial q_j}(q) = \sum_{i,j=1}^{m} \frac{\partial M_{ki}}{\partial q_j}(q)$$

[1] An explicit deduction of the Koszul formula can be found in (Gudmundsson 2018).

holds, or equivalently

$$\sum_{i,j=1}^{m} \frac{\partial M_{kj}}{\partial q_i}(q) = \frac{1}{2} \sum_{i,j=1}^{m} \left[\frac{\partial M_{kj}}{\partial q_i}(q) + \frac{\partial M_{ki}}{\partial q_j}(q) \right]. \tag{B.17}$$

Substitution of (B.17) in (B.4) lets us to rewrite $h_{kji}(q)$ as

$$h_{kji}(q) = \frac{1}{2} \left[\frac{\partial M_{kj}}{\partial q_i}(q) + \frac{\partial M_{ki}}{\partial q_j}(q) - \frac{\partial M_{ij}}{\partial q_k}(q) \right] = c_{ijk}(q). \tag{B.18}$$

Hence the EL equations (B.3) can be identified with the geodesic equations (B.14). Equivalently, the covariant derivative in (B.12) with the Christoffel symbols given as in (B.15) is written in compact form as

$$\overset{M}{\nabla}_{X(q)} Y(q) = \frac{\partial Y}{\partial q}(q) X(q) + M^{-1}(q) C(q, X(q)) Y(q). \tag{B.19}$$

where $C(\cdot, \cdot)$ is defined in (B.6). If $\dot{q}(t) = X(q(t)) = Y(q(t))$, then clearly

$$\overset{M}{\nabla}_{\dot{q}(t)} \dot{q}(t) = \ddot{q}(t) + M^{-1}(q(t)) C(q(t), \dot{q}(t)) \dot{q}(t) = 0_n. \tag{B.20}$$

In order to incorporate the potential $g(q)$ and external force $B(q)\tau$ in free-coordinate expression (B.20), we consider the forces to be functions from TQ to T^*Q (Bullo and Lewis 2004, Section 4.4). In this case the map is $X \in T_q Q \mapsto M(q)X \in T_q^* Q$, then $M^{-1}(q)g(q), M^{-1}(q)B(q)\tau \in T_q Q$. Recall from (B.2) that $g(q)$ is a potential force, then $g(q) = dP(q)$ (differential of $P(q)$); when passing the differential of $P(q)$ through the mapping $M^{-1}(q)$, we get $M^{-1}(q)(dP(q)) = \text{grad}(P(q))$, the gradient of $P(q)$. Therefore, equation (B.5) can be rewritten in a coordinate-free manner as

$$\overset{M}{\nabla}_{\dot{q}(t)} \dot{q}(t) = -\text{grad}(P(q(t))) + M^{-1}(q(t)) B(q(t)) \tau. \tag{B.21}$$

B.1.2 Structure of $C(q, X(q))Y(q)$

Recall that the ℓ-th component of (B.19) is given by

$$(\overset{M}{\nabla}_{X(q)} Y(q))_\ell = \sum_{j=1}^{n} \frac{\partial Y_\ell}{\partial q_j}(q) X_j(q) + \sum_{k=1}^{n} M_{\ell k}^{-1}(q) \sum_{i,j=1}^{n} c_{ijk}(q) X_i(q) Y_j(q), \tag{B.22}$$

for $\ell \in \{1, \ldots, n\}$. We point out that neither the first nor the second terms of the sum in right hand side of (B.22) have coordinate-free meaning, but only together. Nevertheless, the second term of the sum exhibits structural properties in local coordinates. To see this, we first reformulate the definition of the Coriolis matrix in (B.6) as follows: $C(q, X)$ is any matrix whose (j, k)-th entry satisfies the identity

$$C_{jk}(q, X(q))Y_j(q) := \sum_{i=1}^{n} c_{ijk}(q)X_i(q)Y_j(q), \tag{B.23}$$

for any $X(q), Y(q) \in T_q Q$. A natural choice of $C(q, X(q))$ is the following

$$C_{jk}(q, X(q)) = \sum_{i=1}^{n} c_{ijk}(q)X_i(q), \quad \text{for all} \quad j, k \in \{1, \dots, n\}. \tag{B.24}$$

With this choice the matrix defined by $N(q, X) := \dot{M}(q) - 2C(q, X)$, where $\dot{M}(q) = Q(q, X) := \frac{\partial M}{\partial q}(q)X(q) = \sum_{i=1}^{n} \frac{\partial M_{kj}}{\partial q_i} X_i$, satisfies the following:

Corollary B.1. *Let $C(q, X(q))$ defined as in* (B.24). *Then, matrix $N(q, X(q))$ is linear in $X(q)$ and skew-symmetric for every $q \in Q$ and any $X(q) \in \mathfrak{X}^{\infty}(Q)$.*

Proof: The (k, j)-th entry of the matrix $N(q, X(q))$ is given by

$$\begin{aligned} N_{kj}(q, X(q)) &= \sum_{i=1}^{n} \left\{ \frac{\partial M_{kj}}{\partial q_i} - \left[\frac{\partial M_{kj}}{\partial q_i} + \frac{\partial M_{ki}}{\partial q_j} - \frac{\partial M_{ij}}{\partial q_k} \right] \right\} X_i(q), \\ &= \sum_{i=1}^{n} \left\{ \frac{\partial M_{ij}}{\partial q_k}(q) - \frac{\partial M_{ki}}{\partial q_j}(q) \right\} X_i(q). \end{aligned}$$

Linearity on $X(q)$ follow directly from the sums properties. By symmetry of $M(q)$, when we interchange k and j, we get $N_{ckj} = -N_{cjk}$. ■

The skew-symmetric property in Corollary B.1 implies that $Y^\top N(q, X)Y = 0$ for all $Y \in \mathfrak{X}^{\infty}(Q)$. To analyze its geometric meaning, first consider the coordinate expression of the left and right hand sides of the compatibility condition (B.10), that is (see (B.19))

$$L_X(M\langle Y, Z\rangle) = \left[Z^\top M(q)\frac{\partial Y}{\partial q} + Y^\top M(q)\frac{\partial Y}{\partial q} \right] X + Y^\top \underbrace{\left[\frac{\partial M}{\partial q} X \right]}_{\dot{M}(q)} Z, \tag{B.25}$$

and

$$M\langle \overset{M}{\nabla}_X Y, Z\rangle + M\langle Y, \overset{M}{\nabla}_X Z\rangle = \left[Z^\top M(q)\frac{\partial Y}{\partial q} + Y^\top M(q)\frac{\partial Y}{\partial q} \right] X + 2Y^\top C(q, X)Z. \tag{B.26}$$

Thus, it is straightforward to see that the result in Corollary B.1 is nothing else that the coordinate expression of the compatibility condition (B.10) for $Z = Y$, than is,

$$L_X(M\langle Y,Y\rangle) - 2M\langle \overset{M}{\nabla}_X Y, Y\rangle = 0 \iff Y^\top \left[\dot{M}(q) - 2C(q,X)\right] Y = 0. \tag{B.27}$$

The skew-symmetry of $N(q, X(q))$ can be equivalently rewritten as follows.

Lemma B.2. *$N(q, X(q))$ is skew-symmetric if and only if*

$$\dot{M}(q) = C^\top(q, X(q)) + C(q, X(q)). \tag{B.28}$$

Proof: Directly from the definition of the skew-symmetric matrix in $N(q, X)$.

A clear geometric understanding of (B.28) is presented in (van der Schaft 2017, Section 4.6) also in terms of condition (B.10) like in Corollary B.1. The time derivative of the (k, j)-th entry of $\dot{M}(q)$ is denoted by $\dot{M}_{kj}(q) = L_X(M_{k,j}(q))$, with $\dot{q} = X(q)$. This fact can be reformulated in terms of condition (B.10) as follows: Let $Y = \frac{\partial}{\partial q_k}$ and $Z = \frac{\partial}{\partial q_j}$ be in the natural basis of $T_q Q$. Then, the left-hand side of (B.10) is

$$L_X M\left\langle \frac{\partial}{\partial q_k}, \frac{\partial}{\partial q_j}\right\rangle = L_X\left(\frac{\partial}{\partial q_k}^\top M(q)\frac{\partial}{\partial q_j}\right) = L_X(M_{kj}(q)) = \dot{M}_{kj}(q).$$

For the right-hand side of (B.10), from expression (B.19) we have that

$$M\left\langle \overset{M}{\nabla}_X \frac{\partial}{\partial q_k}, \frac{\partial}{\partial q_j}\right\rangle + M\left\langle \frac{\partial}{\partial q_k}, \overset{M}{\nabla}_X \frac{\partial}{\partial q_j}\right\rangle = (C(q,X))^\top_{kj} + (C(q,X))_{kj}.$$

Hence, in this case the coordinate expression of the compatibility condition (B.10) for the k-th and j-th vectors of natural basis of $T_q Q$ a long $\dot{q} = X(q)$ is

$$\dot{M}_{kj}(q) = L_X M\left\langle \frac{\partial}{\partial q_k}, \frac{\partial}{\partial q_j}\right\rangle = (C(q,X))^\top_{kj} + (C(q,X))_{kj}, \tag{B.29}$$

for all $k, j \in \{1, \dots, n\}$ or in compact matrix form (B.28). Moreover, the symmetric property of $\Gamma^\ell_{ij}(q)$ in (B.15) together with the torsion-free condition $T(X, Y) = 0$ (B.9) in the Koszul formula (do Carmo 1992) imply that

$$C(q, X(q))Y(q) = C(q, Y(q))X(q), \quad \text{for all} \quad X(q), Y(q) \in \mathfrak{X}^\infty(Q). \tag{B.30}$$

B.1.3 Energy conservation and internal workless forces

For simple mechanical systems, the (co-)energy is a function $\mathcal{E} : TQ \to \mathbb{R}$ given in natural coordinates by $\mathcal{E}(q, v) = \frac{1}{2} v^\top M(q) v + P(q)$. Thus, the time-derivative of the (co-)energy along the curve γ with local representative $q(t)$ results in

$$\begin{aligned}\frac{d\mathcal{E}}{dt}(q(t),\dot{q}(t)) &= M\langle \overset{M}{\nabla}_{\dot{q}(t)}\dot{q}(t),\dot{q}(t)\rangle + \langle dP(q(t),\dot{q}(t)\rangle,\\ &= -M\langle \mathrm{grad}(P(q)),\dot{q}\rangle + M\langle M^{-1}(q)B(q)\tau,\dot{q}\rangle + \langle dP(q),\dot{q}\rangle,\\ &= \langle B(q)\tau,\dot{q}\rangle.\end{aligned} \tag{B.31}$$

Therefore, in the absence of external forces, the (co-)energy is constant along the solutions of (B.21). This shows that (co-)energy conservation is an intrinsic property.

Remark B.3. *An interesting observation derives from taking the derivative of* $E(q(t),\dot{q}(t))$ *along the system in local coordinates* (B.5)*, that is,*

$$\dot{\mathcal{E}}(q(t),\dot{q}(t)) = \dot{q}^\top\tau + \frac{1}{2}\dot{q}^\top\left[\dot{M}(q) - 2C(q,\dot{q})\right]\dot{q} = \dot{q}^\top\tau. \tag{B.32}$$

We see that the energy conservation holds due to the skew-symmetry of $N(q,\dot{q})$ *(see Corollary B.1). In fact, the skew-symmetry of* $N(q,\dot{q})$ *is stronger than the energy-conservation where it is only required that* $\dot{q}^\top\left[\dot{M} - 2C(q,\dot{q})\right]\dot{q} = 0$ *is required only for all* $\dot{q} = X(q) = Y(q)$*, see* (B.27)*, which is a* mathematical *property induced by the Riemannian metric.*

From a physical point of view, Remark B.3 also implies that $F_N(q,\dot{q}) := N(q,\dot{q})\dot{q}$ is a true force, with $\dot{q} = X(q)$; while the $(q,\dot{q})$-parametrized force $F_{Nv}(q,\dot{q},\dot{q}_v) := N(q,\dot{q})\dot{q}_v$ is virtual, with $\dot{q}_v = Y(q)$. The term *virtual force* means that whenever $\dot{q}_v = \dot{q}$, the identity $F_N^v(q,\dot{q},\dot{q}_v) = F_N(q,\dot{q})$ holds. Notice that (B.27) implies that the forces $F_N(q,\dot{q})$ and $F_N^v(q,\dot{q},\dot{q}_v)$ are *workless*; that is, their resulting power $\dot{q}^\top F_N(q,\dot{q})$ and $q_v^\top F_N^v(q,\dot{q},\dot{q}_v)$ is zero, respectively.

Recall the coordinate definition of the Coriolis and Centrifugal forces $C(q,\dot{q})\dot{q}$ in (B.6) and split it as

$$C(q,\dot{q})\dot{q} = F_C^{cc}(q,\dot{q}) + F_C^{gr}(q,\dot{q}), \tag{B.33}$$

where

$$F_C^{cc}(q,\dot{q}) := \frac{1}{2}\dot{M}(q)\dot{q} \quad \text{and} \quad F_C^{gr}(q,\dot{q}) := \frac{1}{2}\dot{M}(q)\dot{q} - \frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^\top M(q)\dot{q}\right). \tag{B.34}$$

In particular, notice that the power of the force $F_C^{gr}(q,\dot{q})$ is given by[2]

$$\dot{q}^\top F_C^{gr}(q,\dot{q}) = \frac{1}{2}\dot{q}^\top\dot{M}(q)\dot{q} - \dot{q}^\top\frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^\top M(q)\dot{q}\right) = 0, \tag{B.35}$$

[2]From the identity $\dot{q}^\top\frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^\top M(q)\dot{q}\right) = \frac{1}{2}\dot{q}^\top\dot{M}(q)\dot{q}$, see (Arimoto and Miyazaki 1984) for details.

that is, $F_C^{gr}(q,\dot q)$ is workless. Thus, from (B.33) the power $\dot q^\top C(q,\dot q)\dot q$ is equivalent to

$$-\dot q^\top N(q,\dot q)\dot q = 2\dot q^\top F_C^{gr}(q,\dot q) = 0 \tag{B.36}$$

which is a necessary condition for the (co-)energy balance in (B.32). Identity (B.36) gives intuitive proof of $\dot q^\top N(q,\dot q)\dot q = 0$ in terms of the workless forces.
Hence, force (B.34) can be written as the sum of a pure Coriolis ($j \neq i$) and centrifugal ($j = i$) forces $F_C^{cc}(q,\dot q) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial M_{kj}}{\partial q_i}\dot q_i \dot q_j$ and of workless force $F_C^{gr}(q,\dot q)$.

According to (Chang 2002), any workless force $F(q,\dot q)$ can be expressed as

$$F(q,\dot q) = S(q,\dot q)\dot q \tag{B.37}$$

for some skew-symmetric matrix $S(q,\dot q)$. Indeed, we have that $F_N(q,\dot q) = S_N(q,\dot q)\dot q$ and $F_{Nv}(q,\dot q,s) = S_N(q,\dot q)s$, with skew-symmetric matrix $S_N(q,\dot q) := N(q,\dot q)$. Similarly, $F_C^{gr}(q,\dot q) = S_L(q,\dot q)\dot q$ where, by using (B.24), the (k,j)-th entry of $S_L(q,\dot q)$ is[3]

$$S_{Lkj}(q,\dot q) = \frac{1}{2}\sum_{i=1}^{n}\left\{\frac{\partial M_{ki}}{\partial q_j}(q) - \frac{\partial M_{ij}}{\partial q_k}(q)\right\}\dot q_i. \tag{B.38}$$

This means that matrix $S_L(q,\dot q)$ is skew-symmetric, homogeneous and linear in $\dot q$. From (B.34) this in turn implies that the following useful identities hold

$$-\frac{\partial}{\partial q}\left(\frac{1}{2}\dot q^\top M(q)\dot q\right) = \left[S_L(q,\dot q) - \frac{1}{2}\dot M(q)\right]\dot q \quad \text{and} \quad N(q,p)\dot q = -2S_L(q,p)\dot q. \tag{B.39}$$

Using identity (B.39), we can rewrite (B.5) as in (Arimoto and Miyazaki 1984), i.e,

$$M(q)\ddot q + \left[\frac{1}{2}\dot M(q) + S_L(q,\dot q)\right]\dot q + g(q) = \tau. \tag{B.40}$$

Remark B.4 (Energy conservation and symplectic structure of TQ)**.** *An alternative approach to analyze the relation between energy conservation and the workless forces can be done by studying the induced geometric structure by the so-called* Lagrangian vector fields *on TQ: The second-order vector field Z is* Lagrangian *on TQ if*

$$\Omega_L(q,\dot q)(Z(q,\dot q), w) = dE(q,\dot q)\cdot w, \quad \textit{for all} \quad (q,\dot q) \in TQ, \tag{B.41}$$

where $w \in T_{(q,\dot q)}(TQ)$, $dE(q,\dot q)$ is the differential of $E(q,\dot q)$, $\Omega_L(q,\dot q)$ is a Lagrangian

[3]Expression (B.38) coincides precisely with the one pointed out first in (Arimoto and Miyazaki 1984) and later independently in (Koditschek 1984).

two-form that defines a symplectic structure on TQ. Indeed, by the chain rule

$$\frac{d}{dt}E(q,\dot{q}) = dE(q,\dot{q}) \cdot Z(q,\dot{q}) = \Omega_L(q,\dot{q})(Z(q,\dot{q}), Z(q,\dot{q})) = 0,$$

due to Ω_L is skew-symmetric by definition. Thus, we claim that the workless nature of the gyroscopic force $F_C^{gr}(q,\dot{q})$ corresponds to Ω_L in canonical coordinates of TQ. For further details, the interested reader is referred to (Marsden and Ratiu 2013, Chapter 7).

B.2 Mechanical port-Hamiltonian systems

Let us now pass to the *Hamiltonian* formulation of mechanical systems. Let us define the *generalized momenta* p for system (B.5) as

$$p := \frac{\partial L}{\partial \dot{q}}(q,\dot{q}), \tag{B.42}$$

and assume that the defined map $\dot{q} \mapsto p$ is invertible for every q. Introduce the Hamiltonian function H as the *Legendre transformation* of $L(q,\dot{q})$, implicitly given by

$$H(q,p) = p^\top \dot{q} - L(q,\dot{q}), \quad p := \frac{\partial L}{\partial \dot{q}}(q,\dot{q}). \tag{B.43}$$

Define the $2n$ dimensional state vector $[q^\top, p^\top]^\top$. This transforms the n second-order equations in (B.5) into $2n$ first-order equations

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0_n \\ B(q) \end{bmatrix} \tau, \\ y &= B^\top(q) \frac{\partial H}{\partial p}(q,p). \end{aligned} \tag{B.44}$$

We call (B.44) *Hamiltonian control system* with input (effort) τ and output (flow) y. We will also refer to (B.44) as a *mechanical port-Hamiltonian system*; see (van der Schaft and Jeltsema 2014) for further details.

Since $\dot{q}$ and p are conjugate variables (see (B.43)), the state space (called the *phase space*) of system (B.44) is the cotangent bundle T^*Q with local coordinates (q,p).

In physical systems, the Hamiltonian function can be interpreted as the total energy of the system. For instance, in a simple mechanical system with Lagrangian function (B.2), the corresponding Hamiltonian function is given by

$$H(q,p) = \frac{1}{2} p^\top M^{-1}(q) p + P(q), \quad \text{with} \quad p = M(q)\dot{q}. \tag{B.45}$$

It immediately follows that the energy balance is

$$\dot{H}(q,p) = \frac{\partial H^\top}{\partial q}(q,p)\dot{q} - \frac{\partial H^\top}{\partial p}(q,p)\dot{p} = \frac{\partial H^\top}{\partial p}(q,p)\tau = y^\top \tau, \tag{B.46}$$

expressing that the increase of energy is equal to the supplied work. In other words, the energy conservation holds. In this case we say that system (B.44) is *lossless*.

B.2.1 Hamilton equations and Poisson geometry

The structure of the Hamiltonian control system (B.44) and its role in the energy conservation (B.46) can be interpreted in a free-coordinate manner within the context of *almost-Poisson manifolds*[4]. See for further details (Maschke et al. 1992, van der Schaft and Maschke 1995b, Marsden and Ratiu 2013, Chang 2002)

Definition B.5. *Let $\mathcal{X}$ be a differentiable manifold. A Poisson structure on $\mathcal{X}$ is a bilinear map $\{\cdot,\cdot\} : C^\infty(\mathcal{X}) \times C^\infty(\mathcal{X}) \to C^\infty(\mathcal{X})$ called the Poisson bracket and is defined by*

$$(F,G) \mapsto \{F,G\}, \quad \textit{with} \quad F,G \in C^\infty(\mathcal{X}), \tag{B.47}$$

which satisfies for any $F,G,H \in C^\infty(\mathcal{X})$ the following properties

$$\{F,G\} = -\{G,F\} \qquad \textit{skew-symmetry}, \tag{B.48a}$$

$$\{FG,H\} = \{F,H\}G + F\{G,H\} \qquad \textit{Leibniz's rule}. \tag{B.48b}$$

$$\{F,\{G,H\}\} + \{G,\{H,F\}\} + \{H,\{F,G\}\} = 0 \qquad \textit{Jacobi identity}, \tag{B.48c}$$

The pair $(\mathcal{X},\{\cdot,\cdot\})$ is called a Poisson Manifold.

Define the map $X_H(x) : C^\infty(\mathcal{X}) \to \mathbb{R}$ for any $H \in C^\infty(\mathcal{X})$ and arbitrary $x \in \mathcal{X}$ as

$$X_H(x)[F] := \{F,H\}(x), \quad \text{for} \quad F \in C^\infty(\mathcal{X}). \tag{B.49}$$

By the bilinearity of the Poisson bracket and by (B.48b) it follows that $X_H(x) \in T_x\mathcal{X}$ for any $x \in \mathcal{X}$. Thus, $X_H \in \mathfrak{X}^\infty(\mathcal{X})$ is a smooth vector field on $\mathcal{X}$, called the *Hamiltonian vector field*, corresponding to Hamiltonian function H and with respect to $\{\cdot,\cdot\}$. Hence, (B.49) is the *Lie derivative* of F along X_H, i.e., $\dot{F} = L_{X_H}F = \{F,H\}$.
This definition that the Hamiltonian is necessarily a *conserved quantity* of X_H. To see this, by the skew-symmetry property (B.48a) we have that

$$\dot{H}(x) = X_H(x)[H](x) = \{H,H\}(x) = 0, \tag{B.50}$$

[4] An alternative intrinsic approach is given by the so-called *symplectic manifolds* which, under some *rank conditions*, are dual to Poisson manifolds.

that is, the Hamiltonian $H(x)$ is constant along the integral curves of $X_H(x)$. Note that this is independent of the Jacobi-identity (B.48b).

In particular, for system (B.44), the phase space $\mathcal{X} = T^*Q$ withu natral coordinates (q, p) is a Poisson manifold with *canonical* Poisson bracket given by

$$\{F, H\}(q, p) = \sum_{i=1}^{n} \left(\frac{\partial H}{\partial p_i} \frac{\partial F}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial F}{\partial p_i} \right)(q, p), \quad F, H \in C^\infty(\mathcal{X}), \tag{B.51}$$

and (B.50) is the coordinate-free proof of the energy balance (B.46) for $\tau = 0$.

For any Poisson manifold $\mathcal{X}$ with local coordinates $x = (x_1, \ldots, x_r)$, the bracket $\{F, H\}(x)$ only depends on the differentials $dF(x), dH(x) \in T_x^*(\mathcal{X})$ since $\{F, H\} = X_H[F] = dF[X_H]$ and $\{F, H\} = -\{H, F\} = -X_F[H] = -dH[X_F]$. Therefore, the mapping $H \in C^\infty(\mathcal{X}) \mapsto X_H(x) \in T_x\mathcal{X}$ for any $x \in \mathcal{X}$, as defined in (B.49), can also be seen as the mapping from $dH(x) \in T_x^*\mathcal{X} \mapsto X_H(x) \in T_x\mathcal{X}$. Thus, there exist functions $J_{ij} \in C^\infty(\mathcal{X}), i, j \in \{1, \ldots, r\}$ such that the bundle mapping is locally given by

$$\{F, H\}(x) = \sum_{i,j=1}^{r} J_{ij}(x) \frac{\partial F}{\partial x_i}(x) \frac{\partial H}{\partial x_j}(x). \tag{B.52}$$

where $J_{ij}(x) = \{x_i, x_j\}$ for $i, j \in \{1, \ldots, r\}$. Furthermore from (B.48a) it follows that $J_{ij}(x) = -J_{ji}(x)$; and from the Jacobi identity (B.48c), by $\{x_i, \{x_j, x_k\}\} + \{x_j, \{x_k, x_i\}\} + \{x_k, \{x_i, x_j\}\} = 0$, is equivalent to the integrability condition

$$\sum_{l=1}^{r} \left(J_{lj} \frac{\partial J_{ik}}{\partial x_l} + J_{li} \frac{\partial J_{kj}}{\partial x_l} + J_{lk} \frac{\partial J_{ji}}{\partial x_l} \right) = 0, \quad i, j, k \in \{1, \ldots, r\}. \tag{B.53}$$

Conversely, if some smooth functions $J_{ij}(x) \in C^\infty(\mathcal{X})$, $i, j \in \{1, \ldots, r\}$ defined locally on $\mathcal{X}$, satisfy above conditions, then one can define locally the Poisson bracket as in (B.52) verifying (B.48a), (B.48c) and (B.48b) and uniquely determined by its *structure matrix* $J(x) = [J_{ij}(x)]$ which is skew-symmetric and satisfies (B.53). In consequence, any constant skew-symmetric $J \in \mathbb{R}^{r \times r}$ defines a Poisson bracket.
The *rank* of a Poisson bracket $\{\cdot, \cdot\}$ at any point $x \in \mathcal{X}$ is defined as the rank of the structure matrix $J(x)$ at this point and its independent of the choice of coordinates. By the skew-symmetric property of $J(x)$, it rank is necessarily *even*[5].
Furthermore, it follows from (B.49) that the Hamiltonian vector field X_H is expressed in

[5] A Poisson manifold $\mathcal{X}$ whose rank is equal everywhere to the dimension of $\mathcal{X}$ is called a *sympletic manifold*. Thus, the dimension of a sympletic manifold is necessarily even (Maschke et al. 1992).

coordinates $(x_1, \ldots, x_r)$ as the column vector

$$X_H(x) = \begin{bmatrix} X_{H1}(x) \\ \vdots \\ X_{Hr}(x) \end{bmatrix} = J(x) \begin{bmatrix} \frac{\partial H}{\partial x_1}(x) \\ \vdots \\ \frac{\partial H}{\partial x_r}(x) \end{bmatrix}. \tag{B.54}$$

Thus, the dynamical system defined by X_H reads in local coordinates as

$$\dot{x} = J(x)\frac{\partial H}{\partial x}(x). \tag{B.55}$$

Indeed, for the canonical Poisson bracket (B.51) the structure matrix is

$$J(q,p) = \begin{bmatrix} 0_n & I_n \\ -I_n & 0_n \end{bmatrix}, \tag{B.56}$$

and the $2n$-dimensional vector field $X_H(q,p)$ corresponding to the Hamiltonian function $H \in C^\infty(\mathcal{X})$ in (B.43) and the canonical bracket (B.51) is given by

$$X_H(q,p) = \begin{bmatrix} \frac{\partial H}{\partial p_1}(q,p) & \ldots & \frac{\partial H}{\partial p_n}(q,p) & -\frac{\partial H}{\partial q_1} & \cdots & -\frac{\partial H}{\partial q_n}(q,p) \end{bmatrix}^\top. \tag{B.57}$$

Also, due to $\dot{F} = \{F, H\} = X_H(x)[F]$, it each component of X_H can be expressed as

$$\dot{q}_i = X_H(q,p)[q_i] = \{q_i, H\}(q,p) = \frac{\partial H}{\partial p_i}(q,p), \tag{B.58a}$$

$$\dot{p}_i = X_H(q,p)[p_i] = \{p_i, H\}(q,p) = -\frac{\partial H}{\partial q_i}(q,p), \quad i \in \{1, \ldots n\}. \tag{B.58b}$$

The above implies that the map $J(x) : T_x^*\mathcal{X} \to T_x\mathcal{X}$ is the matrix representation of the *bundle map*[6] $J^\sharp(x) : T^*\mathcal{X} \to T\mathcal{X}$, called the *sharp map*. This implies that the Poisson bracket induces a skew-symmetric and nondegenerate bilinear form $J : T^*\mathcal{X} \times T^*\mathcal{X} \to \mathbb{R}$ as follows

$$J(dF, dH) = \langle dF, J^\sharp(x) dH \rangle. \tag{B.59}$$

With all the elements above, we can give a free-coordinate definition of a generalized Hamiltonian control system as follows:

$$\begin{aligned} \dot{x} &= J^\sharp(x) dH(x) + \sum_{i=1}^m g_i(x) u_i, \\ y_i &= \langle dH(x), g_i(x) \rangle, \quad i \in \{1, \ldots m\} \end{aligned} \tag{B.60}$$

[6]This map is also the inverse of the so-called *flat map* $J^\flat(x) : T\mathcal{X} \to T^*\mathcal{X}$ which lets us relate Poisson manifolds with *symplectic manifolds*. See (Bullo and Lewis 2004, Marsden and Ratiu 2013) for details.

where $J^\sharp(x)$ is the induced sharp map of a generalized Poisson bracket and $g_i \in \mathfrak{X}^\infty(\mathcal{X})$ are input vector fields[7]. The energy balance can performed with (B.59) by assuming that the right hand side of (B.60) is given by the Hamiltonian vector field X_{H_u} which implicitly defines the forced Hamiltonian H_u. Then,

$$\begin{aligned}\dot{H} &= J(dH, dH_u) = \langle dH, J^\sharp(x)dH_u\rangle = \langle dH, J^\sharp(x)dH + \sum_{i=1}^{m} g_i u_i\rangle,\\ &= J(dH, dH) + \langle dH, \sum_{i=1}^{m} g_i u_i\rangle = \sum_{i=1}^{m}\langle dH, g_i u_i\rangle = \sum_{i=1}^{m} y_i u_i = y^\top u,\end{aligned} \tag{B.61}$$

with $u = [u_1, \ldots, u_m]^\top$ and $y = [y_1, \ldots, y_m]^\top$. System (B.60) is related to the notion of *input-output* Hamiltonian control systems in (Nijmeijer and van der Schaft 1990) under the assumption that $g_1, \ldots, g_m \in \mathrm{Im}J(x)$, that is $g_i = X_{C_i}, \in \{1, \ldots, m\}$. Thus,

$$\begin{aligned}\dot{x} &= J^\sharp(x)dH - \sum_{i=1}^{m} X_{C_i} u_i = X_H - \sum_{i=1}^{m} X_{C_i} u_i,\\ \bar{y} &= \sum_{i=1}^{n} C_i(x)\end{aligned} \tag{B.62}$$

where $C_1, \ldots, C_n \in C^\infty(\mathcal{X})$. Then, the forced Hamiltonian $X_u = X_H - \sum_{i=1}^m C_i u_i$, and the energy balance along (B.62) is

$$\dot{H} = \{H, H_u\} = \{H, H - \sum_{i=1}^{m} C_i u_i\} = -\{H, \sum_{i=1}^{m} X_{C_i} u_i\} = \{\sum_{i=1}^{m} y_i u_i, H\} = \{y^\top u, H\}. \tag{B.63}$$

with $\bar{y} = [\bar{y}_1, \ldots, \bar{y}_m]^\top$. Energy conservation is guaranteed when $\tau = 0$. It is easy to verify that in coordinates $\{\bar{y}^\top \tau, H(x)\} = y^\top \tau$ with $g_i(x) = -X_{C_i}(x), i \in \{1, \ldots, m\}$.
For (B.44) we have $X_{C_i}(x) = \left[0_n, b_i(q)^\top\right]^\top$ with $i \in \{1, \ldots, m\}$ and $b_1, \ldots, b_m \in \mathfrak{X}^\infty(Q)$.

B.2.2 Generalized Hamiltonian systems and energy conservation

We have seen that the Leibniz's rule (B.48b) and the skew-symmetry (B.48a) are the two key properties of the Poisson bracket for the definition of the Hamiltonian vector field X_H, its structure matrix $J(x)$, and the energy conservation. On the other hand, the Jacobi-identity (B.48c) was only used to express the integrability of $J(x)$ in (B.53), which is a property that does not play any role in the previous developments on Hamiltonian sys-

[7] The input vector fields may be incorporated to (B.60) in terms of *lifts* as in (Chang 2002) or in terms of Poisson bracket preserving vector fields as in (Maschke et al. 1992). However, here this is not required.

tems[8].

Thus, we generalize the notion of a Poisson manifold $(\mathcal{X}, \{\cdot,\cdot\})$ in Definition B.3 by relaxing the conditions on the bracket. Specifically, only requiring properties (B.48a) and (B.48b) on the Poisson bracket. In this case we say that the pair $(\mathcal{X}, \{\cdot,\cdot\})$ is an *almost Poisson manifold*. In order to avoid notation confusions, the structure matrix of a standard Poisson bracket will still be denoted by $J_{\text{can}}(x)$, whereas the corresponding one to an almost-Poisson bracket will be denoted by $J_{\text{gen}}(x)$.

For a simple mechanical system defined on the almost-Poisson manifold $\mathcal{X} = T^*Q$ with natural coorcinates (q, p), it was shown in (Chang 2002) that the structure matrix has the form

$$J_{\text{gen}}(q,p) = \begin{bmatrix} 0_n & K(q) \\ -K^\top(q) & S(q,p) \end{bmatrix}, \tag{B.64}$$

where $K(q), S(q,p)$ are $n \times n$ matrices with $n = \dim Q$ and $S(q,p)$ is skew-symmetric. Thus, a generalized Hamiltonian vector field on $\mathcal{X} = T^*Q$ in coordinates is given by

$$\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} = X_H(q,p) = \begin{bmatrix} 0_n & K(q) \\ -K^\top(q) & S(q,p) \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} \tag{B.65}$$

A generalized Hamiltonian control system is thus intrinsically written as

$$\begin{aligned} \dot{x} &= J^\sharp_{gen}(x) dH + \sum_{i=1}^{m} g_i u_i, \\ y_i &= \langle dH, g_i \rangle, \quad i \in \{1, \dots m\}. \end{aligned} \tag{B.66}$$

Energy conservation follows directly as for system (B.60).

B.2.3 ”Workless forces” and generalized Poisson brackets

In (Chang 2002), it was shown that the ”workless forces” associated the generalized Hamiltonian system (B.65) can be expressed as[9]

$$\text{vlift}(F_{gr}) = J^\sharp_{gr}(x) dH \tag{B.67}$$

[8]The importance of the Jacobi-identity comes clear when it comes to find *canonical coordinates* as shown in (van der Schaft and Maschke 1995b, Maschke et al. 1992).

[9]The notation vlift$(\cdot)$ is an operation on vector bundles as shown in (Marsden and Ratiu 2013). However, for the purposes of this work, we take as definition of ”workless forces” on $\mathcal{X} = T^*Q$ to (B.67).

where $J_{gr}^{\sharp}(x)$ is a generalized Poisson bracket whose structure matrix is given by

$$J_{gr}(q,p) = \begin{bmatrix} 0_n & 0_n \\ 0_n & S_{gr}(q,p) \end{bmatrix} \tag{B.68}$$

with $S_{gr}(q,p)$ skew-symmetric. The quotes in "workless" are due to the fact that vlift(F_{gr}) is not a true force. Since in coordinates,

$$\text{vlift}(F_{gr}(q,p)) = \begin{bmatrix} 0_n \\ -S_{gr}(q,p)M^{-1}(q)p \end{bmatrix} \tag{B.69}$$

we see that vlift(F_{gr}) has $2n$ components. However, from a physical point of view, only the components acting on the p-dynamics in (B.65) are true forces. The reason why (B.67) is referred as "workless" force is that it does not change the Hamiltonian

$$\text{vlift}(F_{gr})[H] = \langle dH, J_{gr}^{\sharp}(q,p)dH\rangle = J_{gr}(dH,dH) = 0 \tag{B.70}$$

An interesting fact comes out when we translate the identities (B.39) on TQ to the phase space $\mathcal{X} = T^*Q$ though the transformation (B.43), in coordinates. Indeed, take the partial derivative with respect to q of (B.43). Then, we have

$$\frac{\partial H}{\partial q}(q,p) = -\frac{\partial L}{\partial q}(q,\dot{q}) \iff \frac{\partial}{\partial q}\left(\frac{1}{2}p^\top M^{-1}(q)p\right) = -\frac{\partial}{\partial q}\left(\frac{1}{2}\dot{q}^\top M(q)\dot{q}\right). \tag{B.71}$$

and identities in (B.39) on the phase space coordinates are given by

$$\frac{\partial}{\partial q}\left(\frac{1}{2}\dot{p}^\top M^{-1}(q)p\right) = \underbrace{\left[S_H(q,p) - \frac{1}{2}\dot{M}(q)\right]}_{E(q,p)} M^{-1}(q)p, \tag{B.72}$$

with $S_H(q,p) := S_L(q, M^{-1}(q)p)$ in (B.38), and $N_H(q,p)M^{-1}(q)p = -2S_H(q,p)M^{-1}(q)p$. Notice that matrix $S_H(q,p)$ induces a "workless force" given by vlift(F_H) $= J_{gr}^{\sharp}(x)dH$, with matrix $S_{gr}(q,p) = S_H(q,p)$.

It follows that the differential of (B.45) can be rewritten in coordinates as

$$\begin{bmatrix} \frac{\partial H}{\partial q}(q,p) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} = \begin{bmatrix} I_n & E(q,p) \\ 0_n & I_n \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix}, \tag{B.73}$$

and the Hamiltonian system (B.44) takes the following alternative form

$$
\begin{aligned}
\begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0_n & I_n \\ -I_n & -E(q,p) \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix} + \begin{bmatrix} 0_n \\ B(q) \end{bmatrix} \tau, \\
y_E &= \begin{bmatrix} 0_n & B^\top(q) \end{bmatrix} \begin{bmatrix} \frac{\partial P}{\partial q}(q) \\ \frac{\partial H}{\partial p}(q,p) \end{bmatrix},
\end{aligned}
\tag{B.74}
$$

We refer to (B.74) as a *mechanical port-Hamiltonian-like* system. The structure of system (B.74) lets us to keep the conservation of energy property as in (B.46). Indeed, $\dot{H}(q,p) = y_E^\top \tau = y^\top \tau$, see (B.73). The alternative representation (B.74) can be seen as the Hamiltonian realization of (B.40).

Lemma B.6. *The matrix $E(q,p)$ defined as in* (B.72)*, satisfies*

- $\dot{M}(q) = -[E(q,p) + E^\top(q,p)]$,
- $E^\top(q,p)M^{-1}(q)p = -C(q, M^{-1}(q)p)M^{-1}(q)p.$

Proof: From the definition of $E(q,p)$, take

$$
-E(q,p) - E^\top(q,p) = \left(-S_H(q,p) + \frac{1}{2}\dot{M}(q)\right) + \left(-S_H^\top(q,p) + \frac{1}{2}\dot{M}(q)\right) = \dot{M}(q),
$$

and

$$
E^\top(q,p)M^{-1}p = -\left(S_H(q,p) + \frac{1}{2}\dot{M}(q)\right)M^{-1}p = -C(q, M^{-1}(q)p)M^{-1}(q)p.
$$

■

Remark B.7. *In (Stadlmayr and Schlacher 2008), instead of considering the matrix $E(q,p)$ in* (B.72) *the authors propose an explicit* pseudo-*port-Hamiltonian-like (p-pH) representation of the mechanical EL system* (B.5) *with state $x_q = (q, \dot{q}) \notin T^*\mathcal{Q}$ (this is the reason of the adjective "pseudo"). They find an explicit relation between the interconnection matrix of the p-pH representation and the skew-symmetry of $N(q,\dot{q})$. Although this approach works well for stability analysis, the physical insight and properties of such p-pH model are ambiguous.*
In the same direction, but via the almost Poisson bracket, a Hamiltonian realization for the mechanical EL system (B.5) *is introduced in (Sarras et al. 2012); this is performed through the partial change of coordinates $\tilde{p} = M^{-1}(q)p$ as in (Venkatraman et al. 2010). The skew-symmetry of $N(q,\dot{q})$ is found to be intrinsically connected to the Poisson bracket.*

We claim that the results in (Sarras et al. 2012) can be interpreted as the coordinate free version of the ones in (Stadlmayr and Schlacher 2008). In both cases, the p-pH model has as state space to $T\mathcal{Q}$*, while the state space of a mechanical pH system is* $T^*\mathcal{Q}$*. The main consequence the stability properties may not be consistent form a physical point of view.*

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Summary

Recent technological developments of smart industries (or industry 4.0), smart cities, advanced logistics, precision systems, biomedical engineering, among others, require a high level of process automation where the motion control systems (or servo-systems) are key for the fulfillment and good performance of the overall process tasks. Common motion control systems that can be found in these kind of high-tech processes are robots manipulators, mobile robots, multi-robot networks, etc.
All of the aforementioned motion control systems are complex nonlinear engineering systems due to their highly coupled, multivariable and multi-domain nature. Under the right assumptions, these servo systems can be modeled as nonlinear mechanical systems for control design purposes. Therefore, the study of advanced control techniques for (networked) nonlinear mechanical systems is a pertinent subject of research.

In this dissertation constructive control methods for solving the trajectory tracking and group coordination problems of nonlinear mechanical systems are proposed. These design methods are based on the concepts of virtual systems, contraction analysis and passivity. Several practical cases are considered in the present work, the tracking control design in the port-Hamiltonian (pH) framework of fully-actuated mechanical systems, flexible-joints robots and marine craft; and the group coordination of networked mechanical systems in the Euler-Lagrange (EL) framework. Both energy based modeling approaches are suitable for control design purposes, since these allow us to have a clear physical understanding of the control schemes.
The performance of the closed-loop fully-actuated system and of flexible joint robot is evaluated experimentally on a robot platform of two degrees of freedom; whereas the performance of the closed-loop marine craft and of the network of mechanical systems is evaluated via simulations.

The proposed tracking and coordination control methods exploit the notion of virtual control system and take advantage of its contractivity and passivity properties. Roughly

speaking, for a given plant, a virtual system can be understood as a system that can produce all plans trajectories, i.e., the plant's behavior is embedded in the virtual one.
For the tracking control design method, the notions of virtual mechanical control systems and of virtual contraction are employed to propose a control design method called virtual contraction based control (v-CBC). The method consists of three main steps: first, we define a virtual system which embeds all the solutions of a given original system; then, as a second step, a controller is designed such that the closed-loop virtual system is contractive and has a desired steady-state behavior; finally, the third step consists of closing the loop of the original system where the control law is given by the virtual systems controller with its virtual state be replaced by the original system's state. The resulting closed-loop virtual mechanical systems exhibit a number of structural properties such as passivity, structure preserving and existence of monotone maps.
On the other hand, for the group coordination case, three distributed control laws for networked EL are presented. The interaction among the EL systems is modeled by a graph consisting of edges and nodes. A strictly passive virtual system is associated to each node, whereas a strictly passive artificial spring system is associated to each edge. With this configuration, the networked EL system possesses a passivity preserving feedback interconnection structure, that together with the strict passivity of both agents and edges dynamics lead to a strictly passive network dynamics. This guarantees exponential stability of the overall network dynamics.

Resumen

Los recientes avances tecnolgicos de industrias inteligentes (o industrias 4.0), ciudades inteligentes, logística avanzada, sistemas alta de presición, ingeniería biomédica, entre otros, requieren de un alto nivel de automatización de procesos, donde los sistemas de control de movimiento (o servo-sistemas) son clave para el cumplimiento y correcto desempeño de todas las tareas del proceso. Algunos ejemplos comunes de sistemas de control de movimiento en este tipo de procesos de alta tenología son manipuladores robóticos, robots móviles, redes multi-robots, etc.
Los servo-sistemas previamente mencionados son sistemas de ingeniera no lineales y complejos debido a su naturaleza altamente acoplada, multivariable y multi-dominio físico. Bajo suposiciones adecuadas, estos sistemas pueden modelarse como sistemas mecánicos no lineales para propsitos de diseño de esquemas de control. Por tanto, el estudio de tecnicas avanzadas de control para (redes de) sistemas mecnicos no lineales es un tema pertinente de investigación.

En esta tesis se proponen métodos de constructivos de control para resolver los problemas de seguimiento de trayectoria y coordinación de grupos de sistemas mecánicos no lineales. Estos métodos de disenño están basados en los conceptos de sistemas virtuales, análisis de contracción y pasividad. En este trabajo se consideran varios ejemplos prácticos, el diseño de control de seguimiento de trayectoria en el enfoque purto-Hamiltoniano (pH) de sistemas mecánicos completamente actuados, robots de uniones flexibles y vehículos marinos; y por otro lado, el control de coordinacíon de movimiento de redes de sistemas mecánicos en el enfoque de Euler-Lagrange (EL). Ambos enfoques de modelado basado en la energía son adeacuados para propósitos de diseño, ya que cons estos los esquemas de control tienen una clara interpreración física.
El desempeño en lazo cerrado de los sistemas completamente actuado y robot de uniones flexibles es evaluado de forma experimental en una plataforma robótica de dos grados de libertad; mientras que el desempeño en lazo cerrado del vehículo marino y los sistemas mecánicos interconectados en red es evaluado via simulaciones.

Samenvatting

Recente technologische ontwikkelingen op het gebied van industrie 4.0, slimme steden, geavanceerde logistiek, precisie systemen en biomedische engineering, vragen, onder andere, om een hoge mate van geautomatiseerde processen waarin de motion control (of servosysteem) van baanbrekend belang is voor het vervullen van de taken n de prestaties van het bovenliggende systeem. Veel voorkomende motion control systemen in dergelijke high-tech processen zijn robot manipulators, mobiele robots, netwerken robots, etc. Al deze systemen worden door hun gekoppelde, multi variabele en multi domein eigenschappen, complexe en niet lineaire engineering systemen.
Onder de juiste voorwaarden kunnen deze servo systemen voor regeltechniek doeleinden worden beschreven als niet lineaire mechanische systemen. Daarom is de studie van geavanceerde regeltechniek methoden voor niet lineaire mechanische (netwerk) systemen een toepasselijk onderwerp voor onderzoek.

In dit onderzoek worden constructieve regeltechniek methoden ontwikkeld voor het oplossen van trajectvolging en cordinatie problemen van niet-lineaire mechanische systemen. De voorgestelde design methoden zijn gebaseerd op virtuele systemen, contractie analyse en passiviteit. Verscheidende praktische applicaties worden onderzocht, waaronder: trajectvolging regelaar design voor volledige aangedreven mechanische systemen, flexibele arm robots en marine voertuigen in het port-Hamiltonian model raamwerk; en cordinatie van een netwerk van mechanische systemen in het Euler-Lagrange (EL) model raamwerk. Beide op energie gebaseerde raamwerken zijn geschikt voor regeltechniek design doeleinden omdat ze een duidelijke fysieke interpretatie meegeven aan de regel algoritmen. Voor volledig aangedreven systemen en de flexibele robot arm worden de prestaties van het geregelde systeem experimenteel gevalueerd op een robot platform met twee vrijheidsgraden. De prestaties van het geregelde marine voertuig en netwerk van mechanische systemen worden via simulaties onderzocht.
De voorgestelde volg- en cordinatie regeltechnieken exploiteren de principes van virtuele regeltechniek systemen en maken gebruik van de contractiviteit en passiviteit van dit

virtuele systeem. Grof gezegd, voor een gegeven plant, kan het virtuele systeem worden begrepen als een systeem die het dynamische gedragingen van de plant kan produceren, i.e., het dynamische gedrag van de plant zit in het gedrag van het virtuele systeem.
Voor het trajectvolging regel algoritme, worden de concepten van virtuele mechanische regelsystemen en virtuele contractie gebruikt om een regeltechniek methode te ontwikkelen die virtual contraction based control (v-CBC) wordt genoemd. Deze methode bevat drie stappen: eerst definiren we een virtueel systeem die alle oplossingen bevat van het originele systeem; dan, als een tweede stap, wordt een regelaar ontworpen zodat het geregelde virtuele systeem contractief is en een gewenst steady state gedrag heeft; als derde en laatste stap, wordt het originele systeem teruggekoppeld met de regelaar van het virtuele systeem wiens virtuele variabelen worden vervangen door de variabelen van het originele systeem. De geregelde virtuele mechanische systemen die hieruit voortkomen hebben een aantal structurele eigenschappen, zoals passiviteit, structuur behouding en het bestaan van monotone mappen. Voor de groep cordinatie problemen, worden drie gedistribueerde regel algoritmes voorgesteld voor EL netwerk systemen. De interacties tussen de EL systemen in het netwerk wordt gemodelleerd door een graaf. Elke knoop van de graaf wordt geassocieerd met een strikt passief virtueel systeem, en elke lijn in de graaf wordt geassocieerd met een strikt passief veer systeem. Met deze configuratie heeft het EL netwerk systeem een passiviteit behoudend terugkoppel interconnectie systeem, die samen met de strikt passieve dynamica van de knopen en lijnen in de graaf leiden tot strikt passieve netwerk dynamica. Dit garandeert exponentile stabiliteit van de complete netwerk dynamica.